Continuing work on optimal trade execution

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June 26 2007

The power-percentile impact model

(Many authors)

If the price at time t is p_t and we sell k shares, then price changes to

$$p_{t+1} = (1 - \alpha k^{\pi})\, p_t$$

where $0 \leq \alpha < 1$, and $0 \leq \pi$ (similar expression for buying shares)

- What is α?
- What is π?
- Some authors: $\alpha = 1$, others $\alpha = .5$, others $\alpha = 0.22$, etc.
- Discrete time vs. continuous time

The power-percentile impact model

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This talk: analyze sensitivity of optimal policies to errors in parameters

- How to come up with error-robust strategies
- Models with histeresis and trade shortfalls
- Competitive models: another party is trading in the same direction

The power-percentile impact model

Tools

- Robust optimization and stochastic programming
- Approximate dynamic programming
- Computational learning theory

Simple experiment

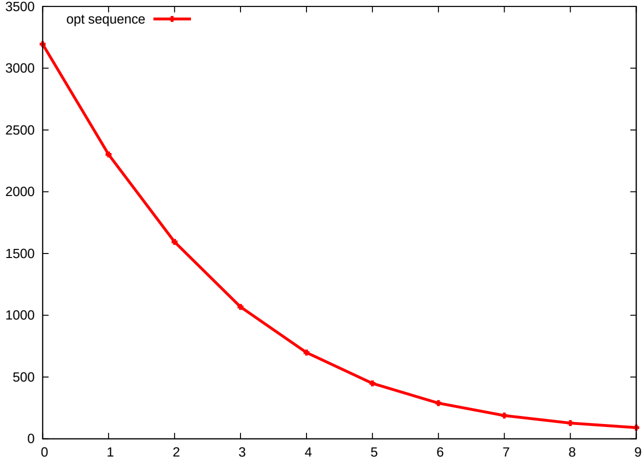
Selling **10000** shares over **10** time periods

$\pi = \mathbf{0.8}$, initial price = **1.0**

α	**total value** (optimal)	**final price** (opt. seq.)
0.00010	8789.77	0.789207
0.00011	8675.20	0.770326
0.00012	8561.75	0.751820
0.00013	8449.43	0.733683
0.00014	8338.22	0.715908
0.00015	8228.10	0.698489
0.00016	8119.08	0.681422
0.00017	8011.14	0.664699
0.00018	7904.27	0.648316
0.00019	7798.45	0.632267
0.00020	7693.69	0.616546

Selling 10000 shares over 10 time periods

$\alpha = \mathbf{0.0001}$, $\pi = \mathbf{0.8}$, initial price = **1.0**: "front loading"



Selling 10000 shares over 10 time periods, $\pi = 0.8$

$\longrightarrow$ Uniform sequence: **robust?**

α	uniform value	opt. value
0.00010	8717.52	8789.77
0.00011	8599.58	8675.24
0.00012	8483.40	8561.94
0.00013	8368.96	8449.87
0.00014	8256.23	8339.05
0.00015	8145.18	8229.48
0.00016	8035.79	8121.17
0.00017	7928.03	8014.12
0.00018	7821.88	7908.33
0.00019	7717.32	7803.81
0.00020	7614.31	7700.57

Robustness: dynamically changing π (α fixed)

Game between 2 parties, a "trader" and "nature":

At time $\mathbf{0}$, the starting value of π, $\pi_{\mathbf{0}}$, is *known* by the trader.

The game ends after $\boldsymbol{T}$ steps, and $\boldsymbol{N}$ shares must be sold altogether.

Notation: price at time $\boldsymbol{t} = \boldsymbol{p_t}$.

1. At time $\boldsymbol{t}$, the trader knows current value of π, $\pi_{\boldsymbol{t}}$, and chooses the amount of shares to sell, $\boldsymbol{k_t}$.
2. Nature then chooses $\pi_{\boldsymbol{t+1}}$.
3. The price changes to $\boldsymbol{p_{t+1}} = \boldsymbol{p_t}(\mathbf{1} - \alpha \boldsymbol{k_t}^{\pi_{t+1}})$, and the trader collects $\boldsymbol{p_{k+1} k_t}$.

Robustness: dynamically changing π

First version: embedded Markov chain

There are n possible values for π: $\pi^{(1)} < \pi^{(2)} < \ldots < \pi^{(n)}$

At time t, assuming $\pi_t = \pi^{(k)}$, then

$$\pi_{t+1} = \begin{cases} \pi^{(k+1)}, & \text{with probability } P_{k,k+1} \\ \pi^{(k)}, & \text{with probability } P_{k,k} \\ \pi^{(k-1)}, & \text{with probability } P_{k,k-1} \end{cases}$$

These probabilities are **known**, $P_{k,k-1} + P_{k,k} + P_{k,k+1} = 1$ and $P_{1,0} = P_{n,n+1} = 0$.

The objective: produce a **policy** with maximum the **expected** total value (or, e.g., value-at-risk).

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Example: Selling 5000 shares over 20 time periods

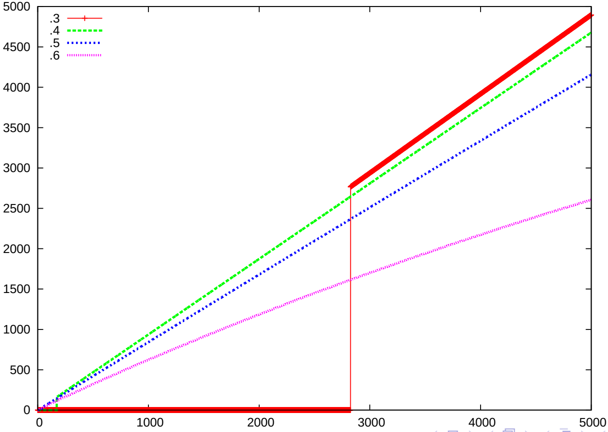
Markov chain:

k	$\pi^{(k)}$	$P_{k,k-1}$	$P_{k,k}$	$P_{k,k+1}$
1	.1	0	.10	.90
2	.2	.10	.20	.70
3	.3	.20	.30	.50
4	.4	.30	.40	.30
5	.5	.25	.50	.25
6	.6	.30	.40	.30
7	.7	.50	.30	.20
8	.8	.70	.20	.10
9	.9	.90	.10	0

Example: Selling 5000 shares over 20 time periods

Optimal policy at time $t = 16$

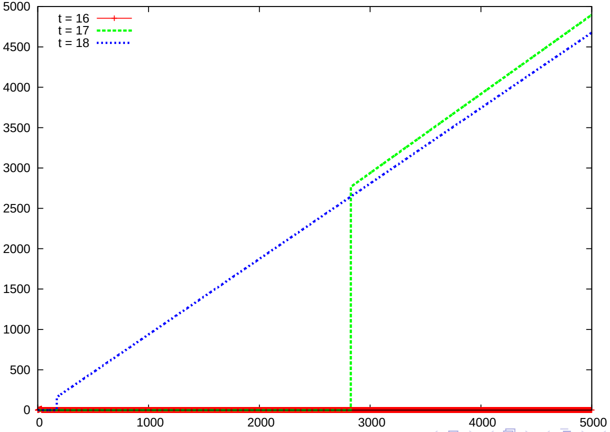
How much to sell as a function of inventory, for several values of π_{16}



Example: Selling 5000 shares over 20 time periods

Optimal policy at time $t = 16, 17, 18$ when $\pi = .7$

Graph shows how much to sell as a function of inventory



Example: Selling 5000 shares over 20 time periods

Robustness studies

Recall: $\pi \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$
with "expectation" $\pi = 0.5$

policy	**markov** optimal	$\pi = $ **0.4** (4850.01)	$\pi = $ **0.5** (4655.28)	$\pi = $ **0.6** (4234.24)	**constant**
mean	4528.79	4270.46	4304.71	4346.80	4076.27
stddev	424.45	1081.92	985.71	816.24	406.89

Example: Selling 5000 shares over 20 time periods

Robustness studies: out-of-sample testing

Recall: $\pi \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$
with "expectation" $\pi = 0.5$

$\longrightarrow$ Assume wrong Markov chain:

k	$\pi^{(k)}$	$P_{k,k-1}$	$P_{k,k}$	$P_{k,k+1}$
1	.4	0	.30	. 70
2	.6	.25	.5	.25
3	.8	.70	.30	0

- Wrong average π
- Narrower range of π

Example: Selling 5000 shares over 20 time periods

Robustness studies: out-of-sample testing with wrong Markov chain

policy	**markov** * (4578.79)	$\pi = 0.4$ (4850.01)	$\pi = 0.5$ (4655.28)	$\pi = 0.6$ (4234.24)	**constant**
mean	3425.74	3585.50	3666.96	3772.48	3508.73
stddev	1616.87	1536.11	1384.41	1102.61	292.55

A different notion of robustness

Game between 2 parties, a "trader" and an "adversary":

At time $\mathbf{0}$, the starting value of π, π_0, is *known* by the trader.

The game ends after T steps, and N shares must be sold altogether.

Notation: price at time $t = p_t$.

1. At time t, the trader knows current value of π, π_t, and chooses the amount of shares to sell, k_t.
2. The adversary chooses π_{t+1} – the goal of the adversary is to *minimize* the payoff to the trader, over the entire horizon.
3. The price changes to $p_{t+1} = p_t(1 - \alpha k_t^{\pi_{t+1}})$, and the trader collects $p_{k+1} k_t$.

A different notion of robustness

Constraining the adversary

There are n possible values for π: $\pi^{(1)} < \pi^{(2)} < \ldots < \pi^{(n)}$
and a parameter $0 \le \lambda \le 1$

At time t,

- If $\pi_t \le \bar{\pi} \left(= \frac{\sum_k \pi^{(k)}}{n} \right)$,
  then the adversary can choose *any* $\pi^{(k)}$ as π_{t+1},
  so long as $(1-\lambda)\pi_t + \lambda\pi_{t+1}$ "not much more" than $\bar{\pi}$
- If $\pi_t > \bar{\pi}$, then the adversary *must* choose π_{t+1} with
  $(1-\lambda)\pi_t + \lambda\pi_{t+1} \le \bar{\pi}$
  (and so $\pi_{t+1} < \bar{\pi}$)

Example

Selling 1000 shares in 10 steps, $\alpha = \mathbf{0.0001}$

π values = $0.70, 0.72, 0.74, 0.76, 0.78, 0.80, 0.82, 0.84, 0.86, 0.88, 0.90$

$(\bar{\pi} = \mathbf{0.80}),\ \lambda = \mathbf{0.5}$

essaying strategies against the adversary:

	maxmin	$\pi = \mathbf{0.76}$ (984.12)	$\pi = \mathbf{0.80}$ (980.06)	$\pi = \mathbf{0.84}$ (975.22)	**constant**
value	979.69	974.79	975.69	976.28	975.89
ending price	.9638	.9602	.9593	.9584	.9557

Unrolling the game

$\bar{\pi} = \mathbf{0.80}$, game starts with $\pi = \mathbf{0.80}$

	Adversary vs. 0.80-static		Adversary vs. min-max	
step	trader	adversary	trader	adversary
1	**381**	**.86**	**216**	**.80**
2	**238**	**.72**	**175**	**.80**
3	**148**	**.90**	**141**	**.80**
4	**91**	**.70**	**112**	**.80**
5	**56**	**.90**	**89**	**.80**
6	**35**	**.70**	**71**	**.80**
7	**21**	**.90**	**57**	**.80**
8	**14**	**.70**	**49**	**.80**
9	**9**	**.90**	**46**	**.76**
10	**7**	**.70**	**44**	**.90**

Other topics

- Out-of-sample testing of robust policy
- Comparison with Markov-robust policies
- Histeresis, trade shortfalls, scaling

... No time!

On-going work: competition between two traders

- Executing in the same direction
- Time frames, quantities
- Randomness or not
- What can be observed?

... Next talk!