Optimal liquidation against a Markovian limit order book

Patrick Hewlett

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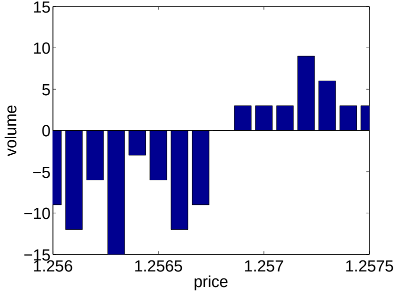
Modelling objectives

- Framework to investigate optimal multiperiod liquidation strategies against a limit order book
- Detailed but tractable stochastic model of transaction costs
  - Spread
  - Temporary price impact
  - Price slippage
- Allowance for risk
- Opportunity costs of delayed trading

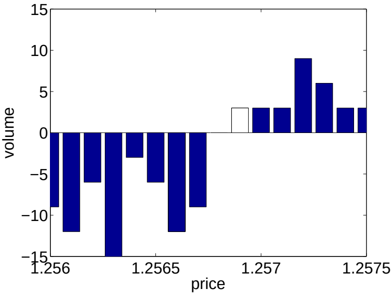
Smith-Farmer limit order book model

Following [SFGK03], we specify a Markov process S_t on the large state space of order books $\mathcal{S}$. Need transition rates for...



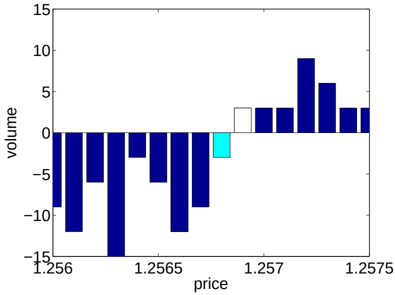
Order book model

Market order arrival: [SFGK03] assume fixed Poisson rate $\mu/2$ for buys and sells



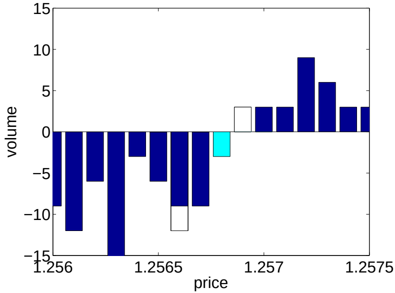
Order book model

Limit order arrival: [SFGK03] assume fixed Poisson rate α per tick below best ask for buy orders, v.v. for sell orders

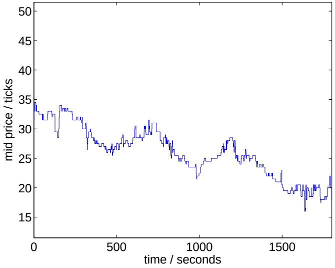


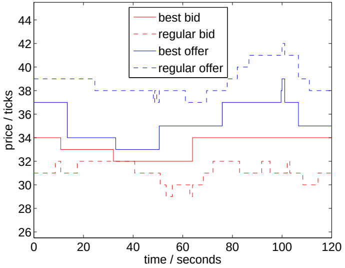
Order book model

Cancellation of existing limit orders: [SFGK03] assume outstanding limit order die at a rate δ



Price and liquidity generated by Smith-Farmer model





- Difference-stationary (log-)prices
- Stationary liquidity, including spread, depth at best etc.
- Non-Markovian mid price

Some extensions

Incorporate the most important "stylised facts of order flow".

- Limit order arrival rates conditional on distance from e.g. best price on same side [MF05]
- Existing limit orders cancelled and immediately resubmitted [Tse06]
- Aggressiveness of orders depends on depth [LS05]
- Fewer market orders when the spread is large [BJP03]
- More limit orders inside spread when depth at best is large [EHJJ03]
- (?Long-range) autocorrelation of signs of consecutive market orders [BGPW03]

Review of Almgren Chriss liquidation model ([AC00])

- Choose deterministic continuous liquidation path $v(t)$ to maximise

$$\mathbb{E}\left(-\exp -\gamma \int_0^T -P^{\mathrm{trans}}(t)v'(t)dt\right)$$

- Equivalent to mean-variance where payoff is Normal
- Linear instantaneous and permanent price impact

$$dP^{\mathrm{mid}} = g(v'(t))dt + \sigma dZ_t$$
$$P^{\mathrm{trans}} = P^{\mathrm{mid}} + h(v'(t))$$

- Speed of trading and hence risk/return controlled via risk aversion parameter γ
- Closed form solution when g linear, and h linear or power law

Critique of Almgren / Chriss at micro level

- What does rate of trading mean in continuous time?
- Functional form of price impact more complex than instantaneous + permanent
- What is the link between parameters g, h of Almgren and Chriss and limit order book dynamics?
- Instantaneous price impact is observably stochastic in many markets
- Can we combine elegant description of risk-return trade-off in Almgren / Chriss with detail of Smith-Farmer type models?

Set-up of optimisation

- Objective (cf. [AC00]) is to sell V_0 units of an asset to maximise risk-adjusted average sale price, using market orders only

$$\sup_{\tau_{V_0} \leq \ldots \leq \tau_1 < T} \mathbb{E}(U(\sum_{i=1}^{V_0} p^{bid}(S_{\tau_i}))) \tag{1}$$

where S_t follows the above dynamics, except at times $\tau_{V_0}, \ldots, \tau_1$, when a market buy order is removed; τ_i adapted to full observation filtration generated by S_t.

- Solution will be a "market timing rule" mapping $(s, v, c, T - t) \in \mathcal{S} \times \{1, \ldots V_0\} \times \mathbb{Z} \times \mathbb{R} \mapsto \{\text{sell 1 unit,..., sell v units, wait}\}$

Bellman equation

- Defining the value function ϕ for the problem in the usual way, we have the following dynamic programming equation

$$\phi(s,v,c,T-t) = max \begin{cases} \phi(TMS(s), v-1, c+p^{\mathrm{bid}}(s), T-t) \\ \mathbb{E}(\phi(S_{t+\delta t}, v, c, T-(t+\delta t))|S_t = s) \end{cases} \tag{2}$$

$$\phi(s,v,c,0) = \begin{cases} U(c),\ v=0 \\ -\infty,\ o/w \end{cases} \tag{3}$$

- Direct solution difficult due to curse of dimensionality

Reduction of the problem

- Put $U(c) = -e^{-\gamma c}$. Then the solution can be written in terms of the order book shape and not its location via $\phi(s, v, c, T-t) = -e^{-\gamma c + v p^{\mathrm{bid}}(s)} \overline{\phi}(\overline{s}, v, T-t)$.
- Note that $\phi(\cdot, v, \cdot, \cdot)$ depends explicitly on $\phi(\cdot, v-1, \cdot, \cdot)$, so we can solve sequentially for increasing v.
- For each v, find optimal strategy mapping $(\overline{s}, T-t) \in \overline{\mathcal{S}} \times [0, T] \mapsto \{\mathrm{sell}, \mathrm{wait}\}$
- May be convenient to let $T \to \infty$, and look for a t-independent solution.
- For each v, find time-stationary optimal strategy mapping $\overline{s} \in \overline{\mathcal{S}} \mapsto \{\mathrm{sell}, \mathrm{wait}\}$

Numerical schemes

- Design low-dimensional approximation architecture for $\overline{\phi}$

$$\overline{\phi}(\overline{s}) \approxeq \psi(\beta, \overline{s}) \tag{4}$$

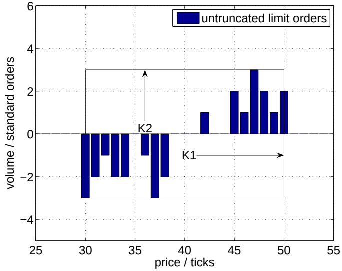
A simple choice (state aggregation) is

$$\overline{\psi}(\beta, \overline{s}) = \sum_i \beta_i \mathbb{I}_{\overline{s} \in \tilde{s}_i} \tag{5}$$

- Tune β so that ψ approximately satifies Bellman equation over $\overline{\mathcal{S}}$, using e.g. approximate value iteration, Q-learning ([TR96], [Ber05])

A particular state aggregation scheme

- In particular, we aggregate by truncating the order book $K1$ ticks from mid and truncating queues at $K2$ orders



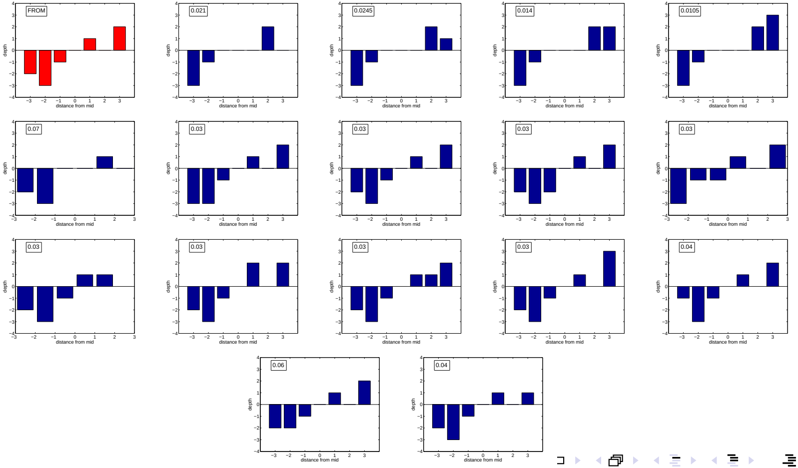
- $K1 = K2 = 3$ results in around 5,000 states

A particular sense of approximate solution of Bellman equation

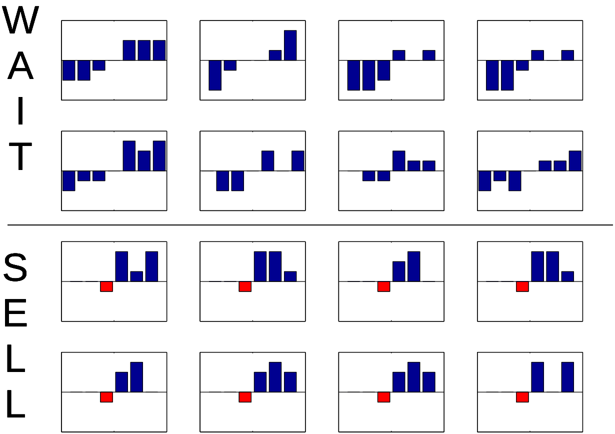
- $\tilde{S}_t$ is non-Markovian, but we can construct an approximating Markov chain $(\tilde{\mathcal{S}}, \tilde{G})$, based on stationary distribution of $\overline{S}_t$.
- (Compute transition rates for the approximating small Markov chain.)
- Solve Bellman equation associated with liquidiation problem on the small chain
- Yields approximate optimal stopping regions for liquidation problem on large chain $\overline{S}_t$, of the form

$$\tau = \inf\{t : (\tilde{S}_t, T - t) \in D\} \quad (6)$$

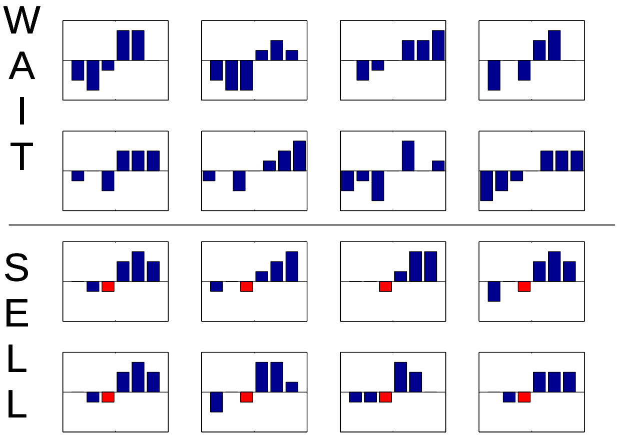
Transitions in approximating OB process



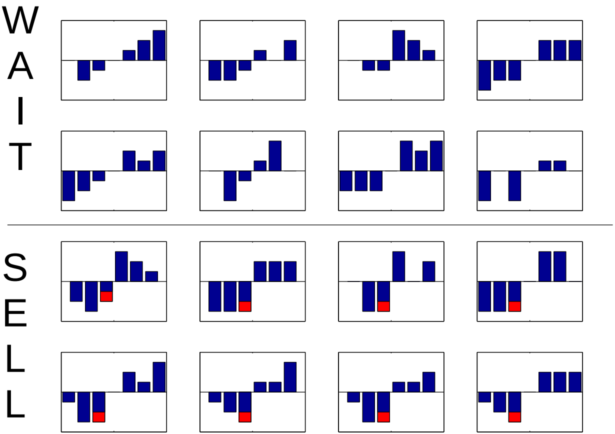
Sell and wait regions, $v = 1$, $T - t = 1000$, $\gamma = 0.00004$



Sell and wait regions, $v = 1$, $T - t = 1000$, $\gamma = 0.04$



Sell and wait regions, $v = 10$, $T - t = 1000$, $\gamma = 0.00004$



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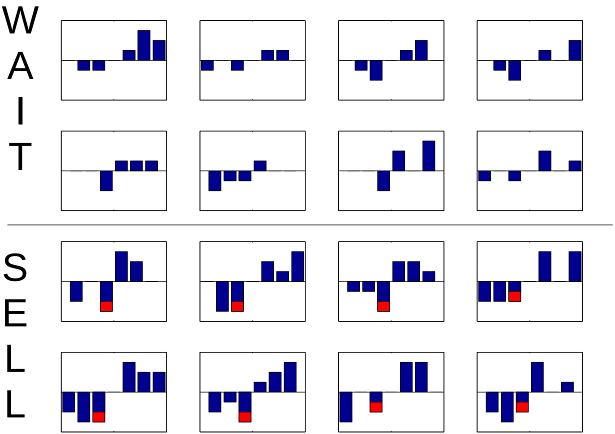
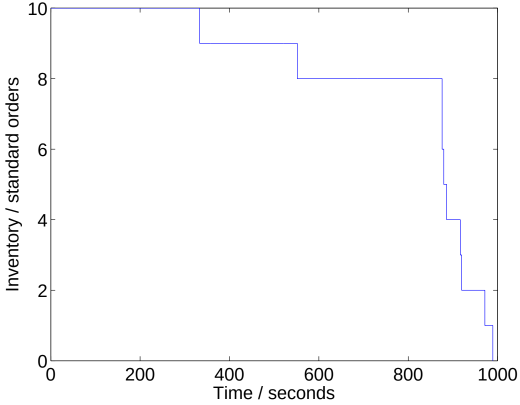
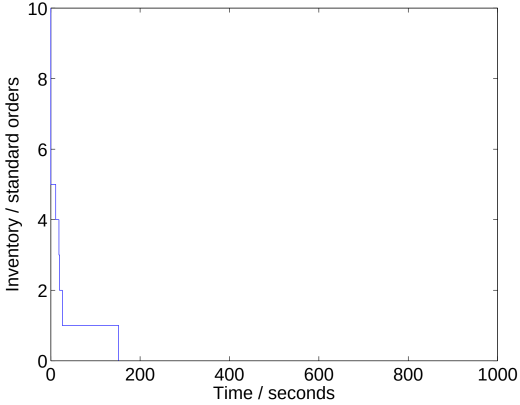


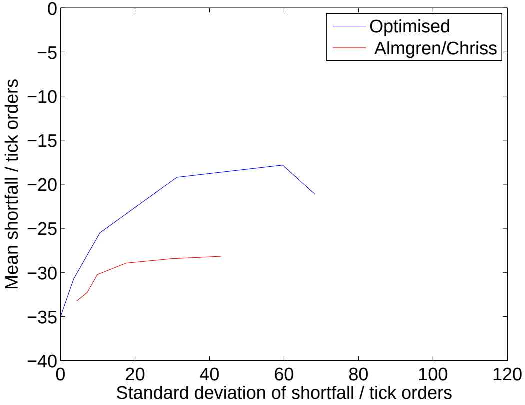
Typical liquidation path, $\gamma = 0.00004$



Typical liquidation path, $\gamma = 0.04$



Liquidation efficient frontier



Outlook

- Further testing against heuristic liquidation strategies
- Improvement and analysis of numerical schemes
- Upper bounds for value function using e.g. duality
- Allow our agent to use limit sell orders
- Robustness to structural assumptions of Smith-Farmer model
- Expand state space to capture salient non-Markovian features of limit order book process
- Partial observation of limit order book state, with application to FX market

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