A Satisficing Alternative to Prospect Theory

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THE INTERFACE OF BEHAVIOURAL FINANCE AND QUANTITATIVE FINANCE
5th Annual CARISMA Conference, London
February 2, 2010

Outline

Motivation

- There are many situations where decision makers possess an *aspiration level* or a *target*:
    - Portfolio managers with compensation schemes based on a benchmark.
    - Individuals with investment goals like retirement or children's education.
    - Consumers looking to make significant purchases or sales may have a reservation price in mind.

Satisficing

- The aspiration level is the core of Simon (1955)'s concept of bounded rationality: Due to
  - limited cognitive resources, and
  - incomplete information

  real-word decision makers may plausibly follow simple heuristics.
- Satisficing behavior says that decision makers accept the first encountered alternative that meets their aspiration levels.

Experimental and field evidence

- Aspiration levels are key drivers of real-word decisions (Lanzillotti 1958, Mao 1970, Payne, Laughhunn, and Crum 1980, 1981).
- Decision makers are highly sensitive to perturbations in the probability of a loss (outcome that fails the aspiration level) or of a gain (outcome that meets the aspiration level); see Payne (2005).
- Consider the lottery

$$(\$100, 0.20; \$50, 0.20; \$0, 0.20; -\$25, 0.20; -\$50, 0.20)$$

Suppose you can add \$28 to one of the outcome. Where would you add them?

Risk aversion or risk seeking?

- Risk aversion is a cornerstone of most theory of choices (e.g., EUT with CRRA, CARA).
- Is risk seeking irrational? This question traces back at least to Friedman and Savage (1948) and Markowitz (1952).
- There is considerable experimental and field evidence that real-world decision makers are risk seeking, particularly for gambles involving losses (e.g., Fishburn and Kochenberger 1979, Hershey and Schoemaker 1980, Payne, Laughhunn, and Crum 1980, Kahnemann and Tversky 1979, and many others)
- Choose between

$$(\$1000, 0.25; \$0, 0.75) \quad \text{and} \quad (\$240, 1).$$

- Choose between

$$(-\$1000, 0.75; \$0, 0.25) \quad \text{and} \quad (-\$250, 1).$$

Our paper

- We propose a model of risky choice that is
    - Simple to motivate
    - Based on the aspiration level
    - “Tolerant” to risk seeking behavior
- We introduce fairly simple properties for preferences
- We derive a representation theorem
- We show that our model is dual to risk measures
- We show that our model can address a number of classical and neo-classical puzzles

Preliminaries (1)

- States described by a set Ω and sigma algebra $\mathcal{F}$
- $\mathcal{V}$ is a set of random variables on Ω (set of "positions")
- τ is a given target and for $V \in \mathcal{V}$, the target premium is $X = V - \tau$; $\mathcal{X} = \{V - \tau \ : \ V \in \mathcal{V}\}$
- A position $X \in \mathcal{X}$ achieves the target iff $X(\omega) \geq 0$
- For $X, \ Y \in \mathcal{X}$, $X \geq Y \Rightarrow X(\omega) \geq Y(\omega)$ for all $\omega \in \Omega$ (e.g., $X \geq 0 \Rightarrow X$ always achieves target)

Preliminaries (2)

- $\succeq$ is a preference relation on $\mathcal{X}$, i.e., for $X, Y \in \mathcal{X}$, $X \succeq Y$ when X is (weakly) preferred to Y.
- As usual, we define $\succ$ by $[X \succ Y \Leftrightarrow (X \succeq Y)\,\text{and}\,\neg(Y \succeq X)]$
- and $\sim$ by $[X \sim Y \Leftrightarrow (X \succeq Y)\,\text{and}\,(Y \succeq X)]$.

Property 1: Some Standard Assumptions and Some Technical Conditions

(i) $\succeq$ is a weak order (complete, transitive, reflexive).

(ii) For $\forall X \in \mathcal{X}$, $\{Y \in \mathcal{X} \;:\; Y \succeq X\}$ is closed in $\mathcal{X}$

(iii) There exists $\mathcal{Z} \subset \mathcal{X}$ that is order-embeddable in $[0,1]$ such that for all $X, Y \in \mathcal{X}$ with $X \succ Y$, there exists $Z \in \mathcal{Z}$ such that $X \succeq Z \succeq Y$.

Property 2: Monotonicity

$$\forall X,\ Y \in \mathcal{X}, \text{ if } X \geq Y, \text{ then } X \succeq Y.$$

Property 3: Satisficing Behavior

$\forall X,\ Y \in \mathcal{X}$:

(i) $X \geq 0 \Rightarrow X \succeq Y$,

(ii) $X < 0 \Rightarrow Y \succeq X$.

Is Diversification Always a Good Thing?

A simple example:

- $|\Omega| = 2$, $X = \{-1, 0\}$, $Y = \{0, -1\}$:
  $\lambda X + (1-\lambda)Y = \{-\lambda, -(1-\lambda)\} < 0$ for $\lambda \in (0,1)$.
- On the other hand,
  $\lambda \cdot (-X) + (1-\lambda) \cdot (-Y) = \{\lambda, 1-\lambda\} > 0$ for $\lambda \in (0,1)$.

Application example: maximizing the Sharpe ratio of investment with respect to a fixed target return τ

- $\max \left\{ \frac{\boldsymbol{\mu}'\mathbf{w} - \tau}{\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}} \; : \; \mathbf{w} \in \mathcal{W} \right\}$
- τ small ⇒ objective quasi-concave
- τ large ⇒ objective quasi-convex

⇒ Treatment of diversification depends somehow on how "safe" or "vulnerable" the target is.

Property 4: Prospective Behavior

There exists a partition of $\mathcal{X}$ into three disjoint subsets $\mathcal{X}_{++}$, $\mathcal{X}_{--}$ and $\mathcal{X}_0$ (secured, vulnerable and neutral) such that
$\forall X_- \in \mathcal{X}_{--},\ X_1,\ X_2 \in \mathcal{X}_0,\ X_+ \in \mathcal{X}_{++}$

$$X_+ \succ X_1 \sim X_2 \succ X_-,$$

and:

(i) $\forall X, Y \in \mathcal{X}_{++}$, if $X \succeq Z,\ Y \succeq Z$ then

$$\lambda X + (1-\lambda) Y \succeq Z \qquad \forall \lambda \in [0,1].$$

(ii) $\forall X, Y \in \mathcal{X}_{--}$, if $Z \succeq X,\ Z \succeq Y$ then

$$Z \succeq \lambda X + (1-\lambda) Y \qquad \forall \lambda \in [0,1].$$

Prospective Satisficing Preference Relation (PSP)

Definition (Prospective Satisficing Preference Relation)

A preference relation $\succeq$ that satisfies Properties 1-4 is called a *prospective satisficing preference relation* (PSP).

Functional Representation of PSPs

Proposition

A preference relation $\succeq$ *is a prospective satisficing preference relation if and only if there exists an upper semi-continuous function* $\rho : \mathcal{X} \to \mathbb{R} \cup \{-\infty, \infty\}$ *such that*

$$X \succeq Y \Leftrightarrow \rho(X) \geq \rho(Y).$$

Properties of ρ

ρ satisfies the following properties for all $X, \ Y \in \mathcal{X}$:

1. *Monotonicity*: If $X \geq Y$, then $\rho(X) \geq \rho(Y)$.
2. *Satisficing behavior*:
   (i) *Attainment content*: If $X \geq 0$, then $\rho(X) = \infty$.
   (ii) *Non-attainment apathy*: If $X < 0$, then $\rho(X) = -\infty$.

Properties of ρ

3. *Prospective behavior*:
   (i) *Superiority of secured premia and inferiority of vulnerable premia*:

$$\rho(X) \begin{cases} > 0 & \text{if } X \in \mathcal{X}_{++}, \\ = 0 & \text{if } X \in \mathcal{X}_0, \\ < 0 & \text{if } X \in \mathcal{X}_{--}. \end{cases}$$

   (ii) *Quasi-concavity (diversifying) over secured target premia*: For all $X, Y \in \mathcal{X}_{++}, \lambda \in [0,1]$:

$$\rho(\lambda X + (1-\lambda)Y) \geq \min(\rho(X), \rho(Y)).$$

   (iii) *Quasi-convexity (concentrating) over vulnerable target premia*: For all $X, Y \in \mathcal{X}_{--}, \lambda \in [0,1]$:

$$\rho(\lambda X + (1-\lambda)Y) \leq \max(\rho(X), \rho(Y)).$$

Prospective Satisficing Measure (PSM)

Definition (Prospective Satisficing Measure)

An upper semi-continuous function $\rho : \mathcal{X} \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ that satisfies properties (1)-(3) in the previous slides is called a *prospective satisficing measure* (PSM).

Given a PSM ρ we can characterize the sets of neutral, secured and vulnerable sets as follows:

$$\begin{aligned} \mathcal{X}_0 &= \{X \in \mathcal{X} \ : \ \rho(X) = 0\} \\ \mathcal{X}_{++} &= \{X \in \mathcal{X} \ : \ \rho(X) > 0\} \\ \mathcal{X}_{--} &= \{X \in \mathcal{X} \ : \ \rho(X) < 0\}\,. \end{aligned} \tag{1}$$

Risk Measures (1)

Definition (Risk Measure)

A function $\mu : \mathcal{X} \to \mathbb{R}$ is a *risk measure* over $\mathcal{X}$ if it satisfies the following for all $X,\ Y \in \mathcal{X}$:

1. $X \geq Y \Rightarrow \mu(X) \leq \mu(Y)$.
2. $\forall c \in \mathbb{R},\ \mu(X + c) = \mu(X) - c$.

Definition (Acceptance Set)

Let μ be a risk measure. The set

$$\mathcal{A}_\mu = \{X \in \mathcal{X} : \mu(X) \leq 0\}$$

is called the acceptance set with respect to μ

Risk Measures (2)

Definition (Convex and Concave Risk Measures)

A risk measure μ is *convex* if, $\forall X,\; Y \in \mathcal{X}$, $\lambda \in [0,1]$:

$$\mu(\lambda X + (1-\lambda)Y) \quad \leq \quad \max\{\mu(X), \mu(Y)\} \qquad \text{(well-studied)}$$

A risk measure μ is *concave* if

$$\mu(\lambda X + (1-\lambda)Y) \quad \geq \quad \min\{\mu(X), \mu(Y)\} \qquad \text{(not well-studied)}$$

We have: If μ is convex risk measure, then $\bar{\mu}(X) = -\mu(-X)$ is a concave risk measure.

Representation Theorem of PSPs by Risk Measures

Theorem

ρ is a prospective satisficing measure if and only if there exists a family of risk measures $\{\mu_k \ : \ k \in (-\infty, \infty)\backslash\{0\}\}$, nondecreasing in k, i.e., $k \mapsto \mu_k(X)$ is nondecreasing for all $X \in \mathcal{X}$, convex if $k \in (0, \infty)$, concave if $k \in (-\infty, 0)$, and with closed acceptance sets $\mathcal{A}_{\mu_k}$ for all $k \in (-\infty, \infty)\backslash\{0\}$, such that

$$\rho(X) \quad = \quad \sup\left\{k \in (-\infty, \infty)\backslash\{0\} : \ \mu_k(X) \leq 0\right\}. \tag{2}$$

(We use the convention $\sup\emptyset = -\infty$). Moreover, given a prospective satisficing measure ρ, the underlying risk measure for $k \in \mathbb{R}\backslash\{0\}$ is given by

$$\mu_k(X) \quad = \quad \inf\{a \ : \ \rho(X + a) \geq k\}. \tag{3}$$

Example: Entropic PSMs (extend Aumann and Serrano 2008's Indexes of Riskiness)

- Consider the family

$$\mu_k(X) = \frac{1}{k} \ln \mathbb{E}\left[\exp(-kX)\right] \qquad k \neq 0.$$

- Associated PSM:

$$\rho(X) \quad = \quad \sup\left\{k \in [-\infty, \infty)\backslash\{0\} \ : \ \frac{1}{k} \ln \mathbb{E}\left[\exp(-kX)\right] \leq 0\right\},$$

  the *entropic prospective satisficing measure* (EPSM).

- If X is normal with (μ, σ^2) then $\bar{\mu}_k(X) = -\mu + k\sigma^2/2$
- In this case, $\rho(X) = 2\mu/\sigma^2$
- Secured (vulnerable) set: RVs with positive (negative) means

Example: CVaR-based PSMs

- Consider the family

$$\mu_k(X) = \begin{cases} \mathrm{CVaR}_{e^{-k}}(X) & \text{if } k > 0 \\ -\mathrm{CVaR}_{e^{k}}(-X) & \text{if } k < 0 \end{cases}$$

- Associated PSM:

$$\rho(X) = \begin{cases} \sup\left\{k > 0 \ : \ \mathrm{CVaR}_{e^{-k}}(X) \le 0\right\} & \text{if } \mathbb{E}[X] \ge 0, \\ \sup\left\{k < 0 \ : \ \mathrm{CVaR}_{e^{k}}(-X) \ge 0\right\} & \text{otherwise,} \end{cases}$$

the *CVaR PSM*.

- If X is normally distributed,

$$\rho(X) = \begin{cases} \sup\left\{k > 0 \ : \ \frac{\phi(\Phi^{-1}(e^{-k}))}{e^{-k}}\sigma(X) \le \mathbb{E}[X]\right\} & \text{if } \mathbb{E}[X] \ge 0, \\ \sup\left\{k < 0 \ : \ \frac{\phi(\Phi^{-1}(e^{k}))}{e^{k}}\sigma(X) \le -\mathbb{E}[X]\right\} & \text{otherwise.} \end{cases}$$

Definition of Stochastic Dominance

Definition (First and Second Order Stochastic Dominance)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

- $X \succeq_{(1)} Y$ (FSD) iff $\mathbb{E}\left[u(X)\right] \geq \mathbb{E}\left[u(Y)\right]$ for all nondecreasing utility functions u
- $X \succeq_{(2)} Y$ (SSD) (resp., RSSD) iff $\mathbb{E}\left[u(X)\right] \geq \mathbb{E}\left[u(Y)\right]$ for all nondecreasing and concave (respectively, convex) utility functions u

Stochastic Dominance Properties of PSMs

Proposition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be atomless. If ρ is a law-invariant PSM, then ρ preserves FSD on $\mathcal{X}$, SSD on $\mathcal{X}_{++}$ (secured positions) and RSSD on $\mathcal{X}_{--}$ (vulnerable positions).

Theorem

Let ρ be a PSM with the following: for $k > 0$, μ_k is bounded from below by the expectation and for $k < 0$, μ_k is bounded from above by the expectation. Then

$$\begin{aligned} \mathbb{E}[X] < 0 \quad &\Rightarrow \quad \rho(X) \leq 0 \\ \mathbb{E}[X] \geq 0 \quad &\Rightarrow \quad \rho(X) \geq 0. \end{aligned}$$

PSPs and the "Allais paradox"

Consider the following two sets of gambles:

- *Gamble A:* Wins \$500,000 for sure.
- *Gamble B:* 1% chance of 0, 10% chance of winning \$2,500,000 and 89% chance of winning \$500,000.

And:

- *Gamble C:* 90% chance of 0, 10% chance of winning \$2,500,000.
- *Gamble D:* 89% chance of 0, 11% chance of winning \$500,000.

Typical observation: A over B; C over D

Evaluating Allais Gambles Using PSPs

τ	Gamble A	Gamble B	Gamble C	Gamble D
55,000	✓		✓	
250,000	✓		✓	
500,000	✓		✓	
695,000		✓	✓	
2,000,000		✓	✓	

- Allais “resolved” for $\tau \in [\$55k, \$500k]$
- True of more general PSMs and also for more general “Allais”-like gamble pairs

PSPs and the “Ellsberg paradox”

Ellsberg’s two-color experiment: Box 1 with 50 red balls and 50 blue balls; Box 2 with red and blue balls in unknown proportions.

First choice:

- *Gamble A:* Win $100 if ball drawn from Box 1 is red.
- *Gamble B:* Win $100 if ball drawn from Box 2 is red.

Second choice:

- *Gamble C:* Win $100 if ball drawn from Box 1 is blue.
- *Gamble D:* Win $100 if ball drawn from Box 2 is blue.

Typical observation: A over B; C over D (though some reverse both)

PSPs and Ambiguity

- Risk measures naturally extend to sets of distributions
- $k > 0$:

$$\mu_k(X) = \sup_{\mathbb{Q}\in\mathcal{Q}} \mu_{\mathbb{Q},k}(X),$$

- $k < 0$:

$$\bar{\mu}_k(X) = \inf_{\mathbb{Q}\in\mathcal{Q}} \bar{\mu}_{\mathbb{Q},k}(X),$$

- Under mild conditions on ρ, the following holds:

$$\begin{array}{lcl} \exists\mathbb{Q} \in \mathcal{Q} \ : \ \mathbb{E}_{\mathbb{Q}}\left[X\right] < 0 & \Rightarrow & \rho(X) \leq 0 \\ \exists\mathbb{Q} \in \mathcal{Q} \ : \ \mathbb{E}_{\mathbb{Q}}\left[X\right] \geq 0 & \Rightarrow & \rho(X) \geq 0. \end{array}$$

Evaluating Ellsberg Gambles Using SPs

Under mild conditions on ρ:

- $$\rho(X_B - \tau) = \rho(X_D - \tau) = 0$$
  for any $\tau \in (0, \$100]$.
- The sign of $\rho(X_A - \tau)$ and $\rho(X_C - \tau)$ depends on whether τ is above or below \$50, i.e., $\rho(X_A - \tau) = \rho(X_A - \tau) > 0$ if $\tau < \$50$, and $\rho(X_A - \tau) = \rho(X_A - \tau) < 0$ if $\tau > \$50$.
- For $\tau \in [0, \$50)$, $A \succ B$ and $C \succ D$, consistently with Ellsberg observations.

Summary

Have developed a new model of choice under uncertainty (PSPs):

- Parameterized by targets/aspiration levels, not “tolerance” levels
- Allows the decision makers to be partially risk-seeking
- Representable via convex/concave risk measures
- Does not require a specific distribution (ambiguity/robustness)
- Can address Allais/Ellsberg and other neo-classical (e.g., Wu and Markle 2008, Machina 2009) “paradoxes” of decision theory

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