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## **PORTFOLIO OPTIMISATION**

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# Portfolio Optimisation

A Status Review –

November 2001

This white paper comprises two parts:

## Portfolio Optimisation (Part I)

*The rise of Markowitz Mean-Variance*

*Models and Beyond*

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## Portfolio Optimisation (Part II)

*The Markowitz Mean-Variance Model:*

*A Technical Perspective*

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## Scope

This white paper introduces Markowitz mean-variance model with a general overview and sets out to explain why and how the finance industry has fully embraced this as method of choice for portfolio planning.

The main focus of the white paper is to bring out many aspects of the portfolio planning problem which are addressed by enhanced mean-variance models that meet the growing requirements of the finance industry.

Portfolio analysis is a leading issue with fund managers who apply such models in many situations such as index tracking, performance evaluation and historical data/backtesting.

The technical underpinning of these methods are described in Part II. A number of currently available software systems are also reviewed together with a broad overview of usage of these systems in Part I.

This white paper will be of interest to:

- Fund managers
  - Trading desk staff
  - Back office staff
  - Quantitative analysts who wish to know the general development in the market place.
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## Portfolio Optimisation (Part I)

In the last few years, the investment community has experienced revolutionary changes on many fronts. It has been felt more so in the midst of the portfolio fund managers, hedge funds, and wealthy tycoons-though it is now spreading even to the common private investor. In this article, we discuss the changing methodologies and technologies finding their way into many aspects of the fund management industry of today.

So what are the underpinning reasons that are bringing an about-face in the day-to-day life of the industry? In a world where competition is the buzzword and volatility omni-present in the financial industry, a reasonable level of returns is a task difficult to be achieved without prior planning and evaluation of investment alternatives. The emergence of progressively sophisticated technologies has hastened the 'modernisation' of the financial industry. So how has the market reacted to the new wave of technology available to it? If the number of operators marketing their services is a benchmark to measure the popularity of optimisation software, then we could safely say that they have become a weapon of choice in the fund-managers' arsenal.

In the next sections, we first portray Professor Harry Markowitz's pioneering work on modern portfolio theory. We follow this up by a road-map of the inroads made by the theory of Markowitz. Then we briefly give an overview on the several models used in the financial marketplace such as CAPM, APT and QP models. Finally we sum up with the main optimisation software currently available in the financial software marketplace.

### An overview of H. Markowitz work on Portfolio Theory

Modern Portfolio Theory (MPT) was introduced by Harry Markowitz with his paper "Portfolio Selection" which appeared in the 1952 Journal of Finance. Thirty-eight years later, he shared a Nobel Prize with Merton Miller and William Sharpe for what has become a broad theory for portfolio selection and corporate finance. It has to be stressed that Harry's theory was scantily used until the late 80's when its popularity grew.

MPT explores how risk-averse investors can construct optimal portfolios taking into consideration the trade-off between market risk and expected returns. The theory quantifies the benefits of diversification. Out of a universe of risky assets, an efficient frontier of optimal portfolios can be constructed. Each portfolio on the efficient frontier offers the maximum possible expected return for a given level of risk.

Investors should hold one of the optimal portfolios on the efficient frontier and adjust their total market risk by leveraging or deleveraging that portfolio with positions in the risk-free asset such as government bonds.

MPT provides a broad context for understanding the interactions of systematic risk and reward. It has profoundly shaped how institutional portfolios are managed, and motivated the use of passive investment management strategies. The mathematics of MPT is also used extensively in financial risk management.

Also, note that the Markowitz model is a single-period approach. It is assumed that an investor has a given initial endowment to invest. The investment will be held for a specific length of time referred to as the investor's holding period. At the end of that period, the investor will liquidate his holdings and will either re-invest it or use it for his own consumption needs (or a combination of both). That is otherwise known as a fixed mix or a buy-and-hold strategy. Hence, we can write the equation of the holding-period rate of return on a security as:

$$\text{Return} = (\text{end-of-period accumulated wealth} - \text{starting-period wealth}) / \text{starting-period wealth}$$

### Asset-Pricing Models

Markowitz work sparked off further research in the area of asset pricing models in finance. We consider two well established models; the Capital Asset Pricing Model (CAPM), and the Arbitrage Pricing Theory (APT).

The Capital Asset Pricing Model (CAPM) was first introduced by William Sharpe in 1964. It extended Modern Portfolio Theory to include the notions of

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systematic and specific risk. CAPM considers a simplified world where:

- There are no taxes or transaction costs.
- All investors have identical investment horizons.
- All investors have identical perceptions regarding the expected returns, volatilities and correlations of available risky investments.

In such a simple world, Tobin's super-efficient portfolio (see Capital Market Line) must be the market portfolio. All investors will hold the market portfolio, leveraging or deleveraging it with positions in the risk-free asset.

CAPM divides the risk of holding risky assets into systematic and specific risk. Systematic risk is the risk of holding the market portfolio. As the market moves, each individual asset is more or less affected. To the extent that any asset is affected by such general market moves, that asset entails systematic risk. Specific risk is the risk, which is unique to an individual asset. It represents the component of an asset's volatility, which is uncorrelated with general market moves.

### **Total Risk= Systematic Risk + Specific Risk**

According to CAPM, the marketplace compensates investors for taking systematic risk, but not for taking specific risk. This is because specific risk can be tackled through diversification. When an investor holds the market portfolio, each individual asset in that portfolio entails specific risk, but through diversification, the investor's net exposure is just the systematic risk of the market portfolio.

Systematic risk can be measured using a parameter beta- which measures the risk of a specific stock relative to the market as a whole. The expected excess return of a portfolio above the risk-free rate is just the portfolio's beta multiplied by the expected excess return of the market portfolio.

The Arbitrage Pricing Theory (APT) is a model of financial instruments and portfolio behaviour based on the proposition that if the returns of a portfolio of assets can be described by a factor structure or model, the expected return of each asset in the portfolio can

be described by a linear combination of the factors with the returns of the asset. The factors can be statistical artifacts; they can be market or industry related; or they can be macroeconomic variables such as interest rates, inflation, industrial production, etc. The resulting factor model can be used to create portfolios that track a market index, to estimate and monitor the risk of an asset allocation strategy, or to estimate the likely response of a portfolio to economic developments. Starting from an initial model proposed by Stephen Ross, APT models have been created for applications in most cash and derivatives markets.

### **What portfolio optimisation software are available**

A number of portfolio optimisation software are available for a quantitative analyst intent on using the latest technologies as an aid for decision-making. The following is a representative list, though by no means exhaustive.

We will start with **Lattice Financial**. The company tries to render its optimisation software more realistic. It includes some business realities into the software--- such as taxes, legal requirements and variable transaction costs.

Lattice also claims its software is advanced as it allows someone to enter the realm of non-convex problems once all important factors have been modelled. There is the availability of Dynamic Optimisation Tools, which the company claims to be a sophisticated optimisation technology that utilises a combination of optimisation algorithms to interact with a simulation model. Other key features of the Dynamic Optimization Tools are:

- They can be integrated into other systems,
- Their high-speed and accuracy,
- They cover a wide range of applications, and,
- They solve **non-linear, nonconvex problems**, as well as **integer problems**.

The second software on offer is the Wagner Math Finance software. WAoptimize is the personal version of WMF portfolio optimisation software. It provides the individual investor with sophisticated

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analytics for rebalancing their portfolio. There are two key quantities every investor seeks to control—return and risk, and we know that the key to controlling risk is diversification. But it is difficult to reduce your risk while maintaining the same expected return. WAoptimize was designed specifically to address this asset allocation problem.

The user interface is a Microsoft Excel workbook, while a customized dynamic link library (DLL) based on WMF versatile MVO Library performs the portfolio optimisation. Advanced features include additional linear constraints and hedging strategies. The most common use of linear constraints is for placing upper and lower bounds on holdings for a group of assets; for example, you want no more than 30% of your portfolio in energy stocks. Wagner offers Waoptimize-C to meet this need. Investors/employees who wish to hedge their positions—for e.g. as a result of owning share options from their companies by using options, could make use of Wagner's Waoptimize-H to address this specialized problem.

Below is a list of standard features for Waoptimize. Some of the enhanced features listed here are standard for M-V Optimizer, Wagner's other Microsoft Excel-based optimizer. WAoptimize can be installed on the Windows system. The features are:

- Handles up to 50 assets/asset classes
- User specified expected returns and volatilities
- User specified correlation matrix
- Displays entire efficient frontier
- Handles long and short positions in assets
- User specified asset-specific minimum and maximum holding constraints
- User specified benchmark portfolio for comparison purposes
- Computes optimal portfolio from a user specified target return
- Computes optimal portfolio from a user specified target volatility
- Computes optimal Sharpe Ratio portfolio.

There are also some advanced features such as:

- Up to 25 general linear constraints (WAoptimize-C)
- More assets/asset classes (up to 250)
- Proportional transaction costs
- Tracking portfolios
- Calculate correlation structure from time series of returns
- Hedging strategies (WAoptimize-H)

The next one we consider is **Finportfolio.com**. It provides guidance through every step of the investment decision-making process with a highly personalized and interactive suite of financial planning and investment analysis tools:

The attributes are:

- FinPortfolio's Strategic Asset Allocation helps you determine optimal mix of assets (such as Large Cap stocks, Small Cap stocks, International stocks, U.S. Bonds and Cash) that will comprise your investment.
- FinPortfolio's results page allows you to interactively change your Initial Investment, Time Horizon, Yearly Contribution, Tax Impact (Taxable, Tax Deferred, or No Tax), and the amount of risk you are willing to take. Your projected wealth and recommended asset allocation will reflect in real-time any changes you make to these assumptions.
- The software's Premium Service allows you to place constraints on asset classes, enabling you to ensure a minimum or maximum weighting of a given asset class within portfolio.
- Moreover, its Portfolio Asset Selection module gives you tremendous power and flexibility in screening prospective portfolio assets, allowing you to select against a single criteria, a benchmark, or a combination of criteria.

- The resulting assets from your asset selection query can be added directly to any of your existing portfolios.
- Quantitative asset selection tools like these cannot be found anywhere else on the Web.
- Portfolio Optimization's highly interactive results page enables you to see, in real-time, the asset allocations necessary to achieve a range of optimal portfolios along the efficient frontier.
- In addition, its Premium Service allows you to place constraints on portfolio assets, giving you an added dimension of control over your portfolio. This is especially important if, for reasons of policy or preference, you want to limit how much or how little of your portfolio a given asset represents.
- FinPortfolio's Premium Service allows you to set alerts. Our automated system will notify you when your portfolio's degree of optimization drops below a certain percentage.
- Also, its **Portfolio Risk Analysis** module calculates the historical average annual return on your portfolio and the variability or uncertainty of that return, and provides you with a risk-adjusted return ratio (the Sharpe ratio) to compare your portfolio to others and to the market.
- FinPortfolio's **Return-based Style Analysis** module informs you to what extent your portfolio is behaving like each of the major asset classes. I.e., it tells you what combination of holdings across various style indexes would have most closely replicated the actual performance of your portfolio.
- While Return-based Style Analysis is typically applied to individual mutual funds, FinPortfolio takes the technique one step further, and applies it to your entire portfolio of mutual funds and stocks.
- FinPortfolio's innovative Asset Analysis module enables you to conduct a quick analysis of assets that you may be considering for inclusion in a portfolio.

- All securities in FinPortfolio's database are updated daily so that you have the most up-to-date information on the risk and return characteristics of each security.

Continuing our survey, we present the APT System produced by **APTLtd** which is essentially an implementation of the Arbitrage Pricing Theory. While the APT theorem (Ross, 1976) focuses on expected asset prices in an arbitrated market, one of its most powerful implications concerns price fluctuations across these assets.

APTLtd provides software and risk models for

- Risk Measurement
- Portfolio optimization
- Risk Attribution
- Performance Analysis

In estimating risk (in the form of covariance matrix) APT uses an algorithm which helps to avoid problems like "concentration ratio" leading to biased estimate of the covariance matrix. These are problems that plague the standard principal component analysis.

They are able to identify risk attribution: shared and specific components.

Products involve:

APTPro 5

which provides a range of portfolio management functions — from risk measurement, through portfolio optimization to risk attribution and performance analysis.

It offers a theory-driven way to attribute risk to multiple types of familiar 'real world' factors — all within the accurate framework of a statistical factor model. By 'mapping' real world variables onto the APT Factors, APTPro is capable of performing the same sort of familiar risk attribution found in fundamental models, but maintaining the accuracy of the APT Factors.

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APTLite is a convenient portfolio management tool for optimization, and risk and performance measurement.

Attributes are:

- Risk Measurement with the highly accurate APT statistical factors
- Portfolio Optimization using APT Factors and the APT Quadratic optimiser
- Performance Analysis assesses how your investment decisions affect your portfolio performance.

APTStore

Is a fully programmable, multi-user database manager for storing and rapidly querying historical financial data. It is easily integrated into existing data systems because of its user-definable import/export standard.

We proceed with the key features of Northfield Information Services Inc. They are as follows:

- It is possible to use any factor of risk with an unlimited number of factors,
- It enables the user to perform many optimisations at a time,
- There is the possibility to incorporate changes in industries, sectors and countries of any security in the data file,
- Can apply arbitrary quadratic penalties and hard constraints on any user-defined descriptors,
- Conforms the optimisation to pre-specified tracking error targets,

The objective function takes lot-by-lot capital gains into account, minimizing taxes while allowing active strategies:

- Can make use of long/short strategies,
- The incorporation of transaction costs and taxes.

We wrap up with the software from **UNICOM**. There is also a brief description and interconnection of part of the work from the paper ‘Computational aspects of alternative portfolio selection models in the presence of discrete asset choice constraints’ (NJ Jobst, MD Horniman, CA Lucas and G Mitra) in the Quantitative Finance Volume I (2001).

The FORTMP/QMIP is designed to use both the interior point method and the sparse simplex (SSX) solution capabilities. Moreover, the system has undergone extensive tests by using QLIB test data (Maros and Mészáros 1997) and models from the finance industry.

To get a realistic view of the portfolio planning problem, some discrete restrictions have been placed-such as:

- (1) A buy-in threshold which basically is the minimum level below which an asset is not purchased,
- (2) The placing of cardinality constraints – where the number of assets to be purchased is specified,
- (3) Making roundlots transactions.

There is an investigation of the shape of the Discrete Constraints Efficient Frontier (DCEF) when incorporating the above constraints. The models are implemented using MPL, a mathematical programming language.

The above authors try to extend the cardinality model to address the portfolio re-balancing problem. Their aim is to identify the trades required to adjust the initial asset holdings such that the optimised portfolio (in terms of variance) tracks a target portfolio or index.

What the authors of the paper above found out is that the use of quadratic mixed-integer problem enabled the use of discrete constraints and thus captured important characteristics of real-world problems. They also highlighted the discontinuities- for DCEF- that arose as a consequence of the imposition of discrete constraints. Moreover, by ‘integer restarting’ the QMIP with the previous solution, they are able to generate a reasonable number of optimal and near optimal points within a restricted branch-and-bound

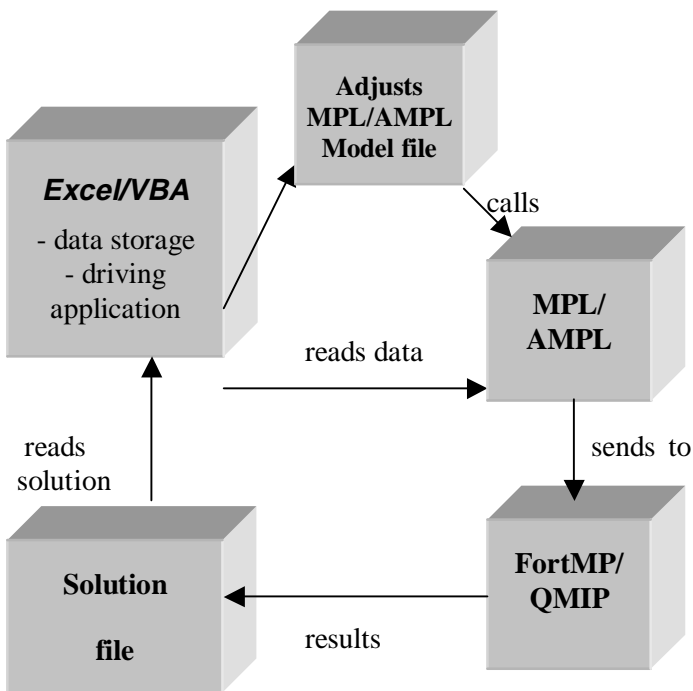
search. They also introduced a ‘reoptimization’ heuristic, which proved to be computationally very efficient in constructing parts of the DCEF.

We now turn on to the many merits of the solution systems:

The solution of convex quadratic programs by SSX (Mitra 1976) and Jones and Mitra (1997) are now deeply-rooted. A branch-and-bound solver, which uses SSX, is by far the most appealing avenue. The other very good features are:

- Speed. In fact, the QMIP problems are solved to the second, improving, feasible integer solution subject to a limit of 500 nodes in the branch-and-bound algorithm.
- Accuracy level. The results are better; for each data set, the mean error is below 0.02% with the median error below 0.015%.
- Scalability. There is the possibility to use a sizeable number of assets in the system-proving in a way its robustness.

Below is a diagram depicting the Data-modelling-solver architecture:



## Concluding Remarks

So are we going to see a frenzy towards these technologically advanced optimisation software? Well, it will take a bit more time for people to realise the help and importance of such software. Furthermore, the success of these optimisation tools depends on the extent to which it replicates real-world situations. Given that investment strategies may be very profitable before taking into account transaction costs and taxation issues, it is very important that these factors are fully incorporated to reflect the situation ‘on the ground’.

One thing is however assured- the genie is definitely out of the box and rapid developments in the field will crop up; with the inevitability that differing software in varying magnitudes will be available for the fund-managers to achieve their investment strategies and goals. Nobody may realise this, but many will be spoilt for choice- in their selection for the software they want to acquire.

## Portfolio Optimisation II

### Introduction and Background

Portfolio Optimisation is the process of analysing a portfolio and managing the assets within it, to obtain the highest return given a level of risk. This paper aims at portfolio managers, fund managers, or individuals wishing to assess the impact of different investment strategies or to actually trace out an investment strategy for a pre-specified investment time horizon.

The basic portfolio optimisation theory hinges on the discrete time, continuous outcome paradigm otherwise known as the mean-variance or Markowitz paradigm. In 1952, Harry Markowitz introduced this approach, which is widely used in applications involving investment portfolios. Mean-variance theory assumes that among portfolios with the same standard deviation, the one with the greatest expected value is the most efficient. Efficient in the sense that for a specified level of expected return, the corresponding risk is minimized; alternatively, for a given level of risk, it yields the highest expected return. He showed how rational investors could build optimal portfolios under conditions of uncertainty by using statistical measures for expectation and variance of return. This set of portfolios is known as the *efficient set* and can be identified by solving a parametric quadratic program. In the risk-return space, the efficient set forms the so-called *efficient frontier*.

However, in the real world investors are interested in extending the basic mean-variance approach with restrictions such as limiting the number of trades, defining a minimum level of trade for an asset, reducing taxation costs, etc. Described portfolio optimisation is static and useful to build an initial portfolio. Over time, the portfolio is rebalanced to justify the actualities of the situation then.

As a technique for evaluating the quality of the portfolio strategy, backtesting is used. The essence of the technique is to compare actual trading results with model-generated measures and help to refine portfolio optimisation techniques.

### Mean-Variance Models

Two mean-variance (MV) models are described below. The first MV model minimizes the risk subject to a desired level of return whereas the second minimizes and maximizes the risk and return respectively. The following MV model determines the efficient portfolio set out of the given investment opportunity set given a specified level of return

#### QP1:

$$\text{Minimize } \sum_{i=1}^n \sum_{j=1}^n x_i x_j \sigma_{ij}$$

$$\text{Subject to } \sum_{i=1}^n r_i x_i = d \quad (1)$$

$$\sum_{i=1}^n x_i = 1 \quad (2)$$

$$x_i \geq 0 \quad \forall i = 1, \dots, n \quad (3)$$

where  $r_i$  defines the expected rate of return of asset  $i$ ,  $d$  the level of return for the portfolio,  $\sigma_{ij}$  the covariance between asset  $i$  and asset  $j$  and  $x_i$  the fraction of the portfolio value invested in asset  $i$ . The objective function minimizes the covariance term, which in turn minimizes the risk of the portfolio. Constraint (1) specifies the return expected from the portfolio and constraint (2) assures that the whole budget is invested. By specifying a level of expected return on the portfolio, the above quadratic model computes the corresponding minimum risk of the portfolio.

The framework is demonstrated in the figure below. Historical data is analysed and reorganized into a collection of analytical data otherwise known as a datamart. The mean-variance model is then built for a

certain desired return level and passed to the optimiser, which works as the planning ‘engine’.

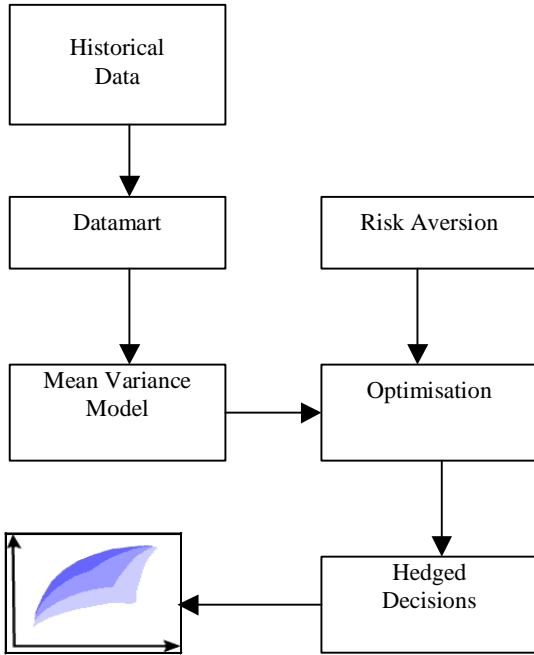


Figure 1: Data flow chart

Solving the model for alternative values of return level  $d$  and plotting the minimum variance against the return gives a curve containing a set of efficient portfolios. This curve is called the *efficient frontier*. These portfolios have the lowest risk for a given level of return.

The second MV model approach may be stated as follows minimizing the risk and maximizing the return:

**QP2:**

$$\text{Minimize } \lambda \sum_{i=1}^n \sum_{j=1}^n x_i x_j \sigma_{ij} - (1 - \lambda) \sum_{i=1}^n r_i^T x_i$$

$$\text{Subject to } \sum_{i=1}^n x_i = 1 \tag{1}$$

$$x_i \geq 0 \quad \forall \quad i = 1, \dots, n \tag{2}$$

where  $\lambda$  is a parameter  $0 < \lambda < 1$ . By varying the parameter  $\lambda$  between zero and one, the efficient frontier is computed.

Unless otherwise stated, we use a dataset consisting of 60 monthly returns for stocks from the FTSE100 to illustrate efficient frontier for the model QP1 described.

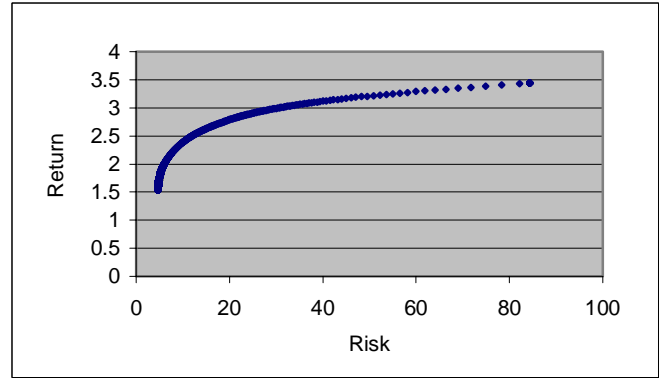


Figure 2: Continuous efficient frontier

**Real World Extensions (1)**

Additional restrictions can be introduced into the general structure to capture some of the real problems faced during portfolio planning.

**Buy-in Threshold Constraint**

$$l_i \delta_i \leq x_i \leq u_i \delta_i \quad i = 1, \dots, n$$

The constraint defines the minimum level at which an asset can be purchased. It eliminates the problem where unrealistically small trades can be included in an optimum portfolio. Adding this constraint changes the efficient frontier as follows

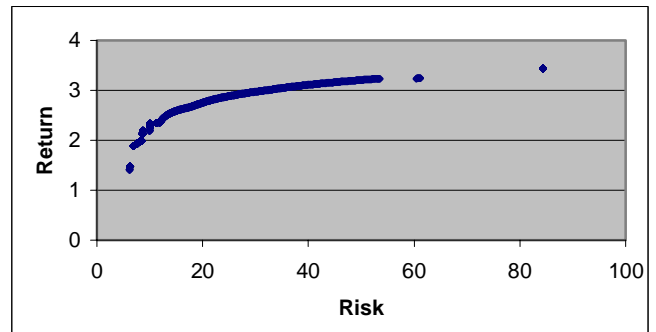


Figure 3: Buy-in efficient frontier

## Cardinality Constraint

$$\sum_{i=1}^n \delta_i = k \quad i = 1, \dots, n$$

Such a constraint restricts the number of stocks allowed in the portfolio and forms the following efficient frontier.

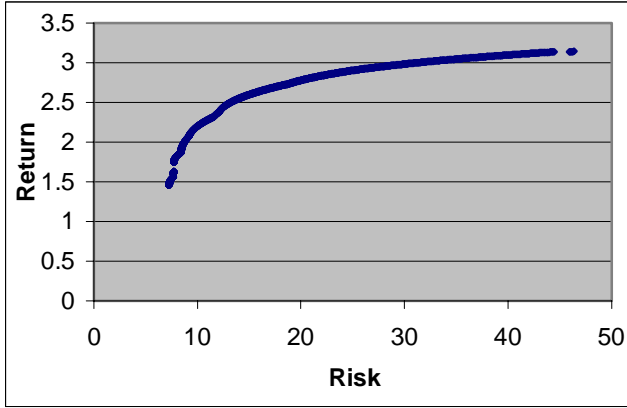


Figure 4: Cardinality efficient frontier

## Roundlots

Roundlots are restrictions used to define the basic unit of investment. Investors are allowed only to make transactions in multiples of these roundlots. When modelling transaction roundlots, the portfolio weights can be represented as

$$x_i = y_i f_i \quad i = 1, \dots, n$$

where  $f_i$  is a fraction of portfolio wealth and  $y_i$  is an integer number of roundlots.

In order to represent *threshold* and *cardinality constraints*, zero-one variables are introduced and to realize *roundlots*, integer variables are required. As a result the QP model becomes a *Quadratic Mixed Integer Programming* (QMIP) model. This increases the size and complexity of the problem. Adding these constraints to the QP model, gives the following QMIP formulation:

$$\begin{aligned} & \text{Minimize} \quad \sum_{i=1}^n \sum_{j=1}^n y_i y_j f_i f_j \sigma_{ij} + \gamma \epsilon^- - \gamma \epsilon^+ \\ & \text{Subject to} \quad \sum_{i=1}^n r_i y_i f_i = d \end{aligned} \quad (1)$$

$$\sum_{i=1}^n y_i f_i + \epsilon^- - \epsilon^+ = 1 \quad (2)$$

$$\begin{aligned} l_i \delta_i &\leq y_i f_i \leq u_i \delta_i \\ \forall i &= 1, \dots, n \end{aligned} \quad (3)$$

$$\sum_{i=1}^n \delta_i = k \quad (4)$$

$$y_i \text{ integer} \quad \forall i = 1, \dots, n \quad (5)$$

$$\delta_i \text{ binary} \quad \forall i = 1, \dots, n \quad (6)$$

$$x_i, \epsilon^-, \epsilon^+ \geq 0 \quad \forall i = 1, \dots, n \quad (7)$$

The efficient frontier is plotted as follows

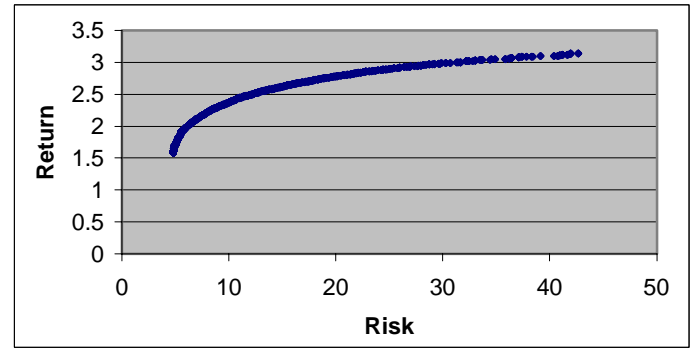


Figure 5: Roundlots efficient frontier

The estimation of the underlying parameters i.e. the expected returns, variances and covariances, is very important. Already small changes in these inputs data can produce a large effect on the optimal portfolio weights (see Chopra and Ziemba 1993). The expected monthly rate of return can be estimated from the mean series of past monthly returns. The covariance between two assets can be estimated on the basis of 'sample covariance' over the same series of returns. The estimation of expected returns as sample estimates based on historical time series data can be unreliable estimates of the future.

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## Real World Extensions(2)

### Sector/Industry (joint) Constraint

A further constraint in the portfolio model is a sector, industry restriction. The set of assets is grouped in various sectors respectively industries which have a minimum and maximum exposure.

$I \in I$  : universe of assets

$I' \in I'_j$  : set of stocks which belong to sector j

*Sector constraint*

$$\sum_{i \in I'_j} x_i \geq L_i$$

$$\sum_{i \in I'_j} x_i \leq U_i$$

Where  $L_j$  denotes the minimum exposure in sector j and  $U_j$  the maximum exposure.

### Immunization – Duration Matching

Immunization is a portfolio strategy that matches interest rate risk of an asset portfolio against the projected stream of liabilities to achieve zero net market exposure.

*Cashflow Matching constraint*

$$\sum_{i=1}^m P_i x_i = P_L$$

With  $P_i$  the portfolio cashflow and  $P_L$  the liability cashflow stream. Under this condition, and assuming reinvestment rates do not change from the assumed values, the portfolio cashflows will match the liability cashflows. There is no guarantee that the necessary condition will hold when the rates change. A first order estimate of the change of the present value of a stream of cashflows, either an asset or liability, with changes in the term structure is measured by duration. Hence, a first-order sufficient condition for portfolio immunization is that the duration of assets and liabilities match.

*Duration Matching constraint*

$$\sum_{i=1}^m D_i^{FW} F_i x_i = D_L^{FW} F_L$$

Where  $D^{DW}$  is the Fisher-Weil duration of the liabilities.

### Taxation

Taxable portfolios represent a special challenge to the investment manager. The imposition of taxes such as capital gains can make many strategies that will be profitable on a pre-tax basis into certain losses on an after-tax basis. Investment managers must simultaneously not only consider the return and risk aspects of each portfolio but also the tax circumstances of the investor.

Strategies to address tax efficiency must have the following properties:

- Reduce the cost of taxes, which is spread between pre-tax and after-tax returns to investors.
- Provide minimum inhibition – (i.e. not limiting turnover) to the portfolio management process, so as to allow active management strategies to be effectively pursued.
- Tax efficient strategies must such that they can be implemented in an automated fashion in order to lessen the burden of handling (potentially) hundreds or thousands of accounts.

A tricky aspect of the taxable account problem is that in reality, it is a multi-period problem, whereas the traditional mean-variance Optimisation looks at the future as a single period. The logic of the tax models is usually captured using discrete 0-1 variables. A transaction amortization factor can be introduced to deal with the multi-period aspect of this process. The choice of the factor will depend on the tax situation as well as the market outlook of the parties involved.

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## Factor Models

To compute the optimal weights for a mean-variance problem, estimates of the expected return and the covariances between the securities in the available population is needed as input. One way of making these estimates is by sampling from past returns. Though this is the most straightforward approach to obtaining these estimates, it has some problems. Sample means of stock returns are unreliable estimates of the true expected return. In order to reduce sampling error, sample estimates require also a lengthy history of past returns. This has the unfortunate consequence that the farther one goes back in time, the more likely the series of the stock returns does not reflect the contemporary character from its past. The firm today may have significantly different character from its past. Accordingly, sample estimates may prove to be unreliable estimates of the future.

Better estimates of covariances and expected returns can be obtained by employing *factor models*. Single factor models are such that the covariance between the returns on stocks can be attributed to a single factor, usually a market index.

In multifactor models, the covariances between the stocks are attributed two or more factors. The factors can be identified as any variable that influences the returns on the stocks. Examples of factors include the market index, growth in industrial production, change in the rate of employment, interest rates or inflation rates.

The variance of returns is made up of two components, the *systematic risk* and *residual variance*. Systematic risk is a risk that cannot be diversified and residual variance represents risk that is unique to an organization and can be eliminated through diversification.

## Capital Asset Pricing Model (CAPM)

The Capital Asset Pricing Model (CAPM) is a prime example of a single factor model, which represents the relationship between a non-diversifiable risk of investments and expected return. The paradigm underlying CAPM is as follows: It looks at the risk and rates of return and compares them to the overall

stock market. The model assumes that most investors are risk adverse and expect to be rewarded.

## Arbitrage Pricing Theory (APT)

The arbitrage pricing theory (APT) was first introduced by Ross in 1976. It focuses on the expected asset returns in an arbitrated market. APT can be more general than the CAPM because it allows for multiple risk factors. Unlike CAPM, APT does not require the identification of the market portfolio. The Factors used are assumed to be both uncorrelated with the specific returns, which are in turn uncorrelated. In other words, we consider several independent risk factors. The theorem shows that an asset's expected return beyond the risk-free rate will simply be the sum of its exposure to some shared sources of risk, weighted by the prices the market assigns to these risks – the risk premia. Under the Arbitrage Pricing theory, market risk is seen as the effect of which is measured by the conventional beta (for stocks) or duration (for bonds).

## Index Tracking

Factor models assume that investors use mean-variance models to build a portfolio such that risk is reduced given a desired level of return. However, no significant funds are actually invested to formally manage this relationship. Some institutions, however, trade expected return against *tracking error* (relative to a benchmark of stock index portfolio). Tracking error is a method of following such an index and the error can be defined as the standard deviation of the difference between the performance of a benchmark given by the index and the planned portfolio. The mean-variance model can be used to track targets. It can find unique portfolios such that they either minimise volatility in residual return given their target beta, maximise correlation with the target, or minimise volatility in the difference between their returns and the target returns.

## Rolling Markowitz

In the real world, portfolios are built such that they can be traded at any time. This creates a situation that is multi-time period. Unfortunately all the models mentioned in the previous section, (CAPM and APT) are single period models. However, they are applied to multi-period situations. When a MV model is solved, the optimal portfolio weights are determined. According to the CAPM theory, these weights should be identical to the market portfolio. If we assume that the statistical properties of the returns from the next time period are equivalent to those from the previous time period, and that both returns are uncorrelated, then the new weightings of the portfolio should equal the previous portfolio weightings. As a result the optimal weights can be carried through to the next time period.

However, as time passes on, the asset prices change, which in turn means the market value changes, since market weights are capitalized weights. The portfolio weights carried forward no longer correspond to the market portfolio weights. This is one of the contradictions of the MV models. A multi-period approach can be developed. However this approach contradicts some assumptions of the single-period model. For example, the multi-period theory prefers price volatility whereas the single-period model does not.

In order to address a multi-period situation, a large number of organizations follow the “Rolling Markowitz” approach over the planning period. This method involves building a portfolio in the initial time period for a given budget. To respond to market changes, the portfolio is re-balanced at regular time intervals.

Figure below shows this approach more clearly. At  $t = t_0=0$ , we compute an initial portfolio. Subsequently for  $t_n$ , ( $n=1, \dots, k$ ) this portfolio is rebalanced.

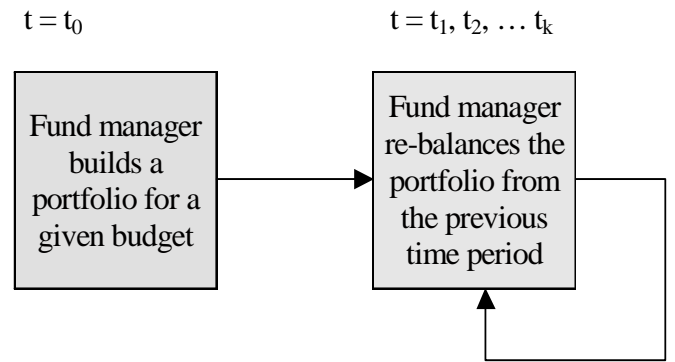


Figure 6: General procedure for re-balancing portfolios

It can be seen from the above diagram that once the fund manager builds a portfolio then he/she will re-balance it at regular time intervals  $t=t_1, t_2, \dots, t_k$ .

The factors may be observable (e.g. changes in inflation) or may be derived through factor analysis (Principal Component Analysis). Implementing factor models can produce better estimates of stock returns and covariance between assets. Another advantage of factor models is that the computational complexity, when estimating portfolio volatility, is reduced.

An algebraic formulation of the factor model developed for tracking a target index is stated below.  $I_c$  defines the degree to which an index is affected by factor  $c$ . A model can be stated which minimizes the difference between the target portfolio and the investor’s portfolio in terms of their risk factors. Such a model can be defined as

### RebalanceQP:

$$\underset{e}{\text{Minimize}} \quad \sum_{c=1}^C y_{P,c}^2 \sigma_{f_c}^2 + \sum_{i=1}^n x_i^2 \sigma_{\varepsilon_i}^2$$

$$\text{Subject to} \quad y_{P,c} = \left( \sum_{i=1}^n x_i \beta_{ic} \right) - I_c \quad \forall \quad c = 1, \dots, C \quad (1)$$

$$\sum_{i=1}^n r_i x_i \geq d \quad (2)$$

$$\sum_{i=1}^n x_i = 1 \quad (3)$$

$$l_i \delta_i \leq x_i \leq u_i \delta_i \quad \forall \quad i=1, \dots, n \quad (4)$$

$$\sum_{i=1}^n \delta_i = k \quad (5)$$

$$x_i \geq 0 \quad \forall \quad i=1, \dots, n \quad (6)$$

$$\delta_i \text{ binary} \quad \forall \quad i=1, \dots, n \quad (7)$$

Where  $y_{p,c}$  is the difference between the portfolio beta and the beta value of the index  $I_c$ . The portfolio beta is the weighted average of the betas (of the assets) in the portfolio with respect to factor  $c$ . By defining this variable, the difference between the factor beta of the portfolio and the beta value of the target portfolio is minimized. In the objective function, the first term represents the risk associated with each factor and the second models the residual variance for the portfolio and summing them together accounts for the total risk of the portfolio.

For the time intervals  $t=t_1, t_2, \dots, t_k$  the portfolio model has the same format as the model *RebalanceQP*, except that it has an extra constraint. This constraint is responsible for the re-balancing activity of the portfolio. It defines the optimal portfolio weights in terms of the initial holdings  $n_i$  the amounts bought  $b_i$  and the amounts sold  $s_i$ . Hence we have

$$x_i = n_i + b_i - s_i \quad i=1, \dots, n$$

In addition, the cardinality and the buy-in constraints are modified accordingly

$$\delta_i^b LB_i^b \leq b_i \leq \delta_i^b UB_i^b \quad \forall \quad i=1, \dots, n \quad (4.1')$$

$$\delta_i^s LB_i^s \leq s_i \leq \delta_i^s UB_i^s \quad \forall \quad i=1, \dots, n \quad (4.2')$$

$$\delta_i^b + \delta_i^s \leq 1 \quad \forall \quad i=1, \dots, n \quad (6')$$

$$\sum_{i=1}^n (\delta_i^b + \delta_i^s) \leq k \quad \forall \quad i=1, \dots, n \quad (7')$$

$$\delta_i^b, \delta_i^s \text{ binary} \quad \forall \quad i=1, \dots, n \quad (9')$$

## Backtesting

Backtesting is a technique whereby a set of trading rules (this could be in the form of a valuation or a forecasting model) is applied to historical stock market data to test the economic impact of following those trading rules. In practice, many investors do not have the time, tools or programming experience to do this. Their decisions are based on highly emotional criteria. However, professional traders study trading patterns over historical time periods and as a result, they can estimate the risk of using a new trading strategy. Various factors can be examined using backtesting, such as earnings, equitable analysis, index tracking, risk/return trade off etc.

Back testing does not, in any manner, imply that history will repeat itself. However people's reactions to the market do not change dramatically over time. If strong earnings reports have made people buy stocks in the past, then they may certainly do so in the future. When we use Markowitz M-V model, backtesting means reapplying the *RebalancingQP* model repeatedly over the historical data stepping through given time intervals.

The circle of the backtesting strategy is shown below.

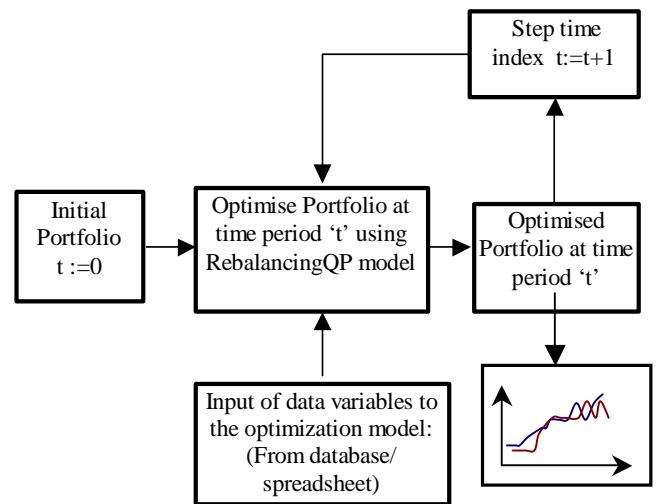


Figure 7 : Backtesting cycle

The representative example performances of the portfolio planning models (see figure below) are compared with an equivalent savings investment and the movement of a market index (S&P 500 stock index), between February 1996 and February 1999.

The results therefore do not fully represent the benefits of re-balancing since it is rather irrational to retrain the same portfolio despite changing market conditions.

The figure also shows that performance of portfolio planning models surpasses the performance of an equivalent savings account investment. The low return on a savings account investment is compensated by negligible risk. This suggests that it may be advantageous to combine a risk-free investment with a (riskier) stock market investment. This underpins the CAPM theory, which encourages investors to seek to attain the market portfolio and some risk free assets.

The continuous re-balancing model slightly outperforms the index for the majority of the investment period. However the index recovers dramatically to beat the portfolio planning models for the last six months. This again is in conformity with the CAPM theory, which maintains that the market portfolio cannot be out-performed on a systematic basis over a period of time.

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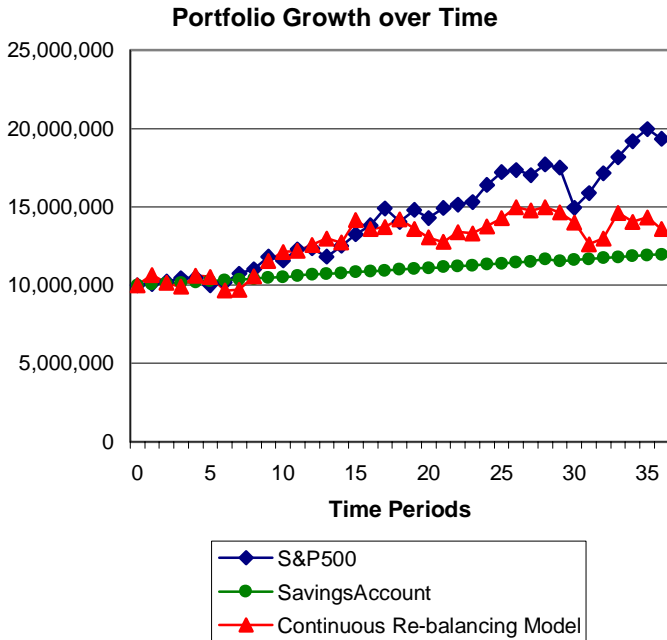


Figure 8: Portfolio growth comparison