MODELS AND SOLUTION METHODS FOR LIABILITY DETERMINED INVESTMENT

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Models and Solution Methods for Liability Determined Investment

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1 Introduction and Background

1.1 Background

Traditional Asset and Liability Management (ALM) models have been recently recast as Liability Driven Investment (LDI) models for making integrated financial decisions in pension schemes investment: matching and outperforming liabilities. LDI has become extremely popular as a decision tool of choice for pension funds. The last decade experienced a fall in the equity markets while bond yields reached low levels. New regulations were introduced whereby liabilities were hard to meet. In the case of a deficit the pension fund trustees and employers have to agree on extra contributions to fill the deficit within 10 years time. The UK Accounting standard FRS17 (since 2001, replacing SSAP24) requires the assets to be measured by their market value and liabilities measured by a projected unit method and a discount rate reflecting the market yields then available on AA rated corporate bonds of appropriate currency and term (see Accounting Standards Board Financial Reporting Standard 17). Furthermore, deficit or surplus has to be fully included on the balance sheet. In the Netherlands and the Nordic countries LDI models have become established; the UK, Italy and few other European countries are close followers of this trend. Traditionally, assets and liabilities were considered separate. In asset management the aim was to maximize return for a given risk level. However, the matching of the liabilities was not taken into consideration. The main argument was that assets should be made to grow faster than liabilities. The modern integrated approach to LDI considers the cash flow streams for invested assets which can be fixed income portfolios enhanced by interest rate swaps and in some cases includes added swaptions.

We present an asset and liability management (ALM) problem for LDI, which we model using three approaches: a deterministic model, a stochastic model incorporating uncertainty and a chance-constrained stochastic model. In the deterministic model we look at the relationship between $PV_{01}$ matching and the required funding. $PV_{01}$ is the change of the net present value of a bond due to 0.01% positive parallel shift in the yield curve. In the stochastic programming model we have
two sources of randomness: liabilities and interest rates. We generate interest rate scenarios, look at the relationship between funding requirements and minimize the deviation of the $PV$ (present value) matching of the assets and liabilities over time.

In the chance-constrained programming model we limit the number of future deficit events by introducing binary variables and a user specified reliability level. The last model has integrated chance constraints, which not only limits the events of underfunding, but also the amount of underfunding relative to the liabilities. Furthermore, a fixed mix model is introduced for testing purposes only.

1.2 Applications of Asset and Liability Management

An integrated asset and liability management (ALM) model sets out to find the optimal investment strategy by considering assets and liabilities simultaneously. The purpose of such an approach is to reduce risk and increase returns; it has been successfully used for banks, pension and insurance companies, university endowments, hedge funds, mutual funds and wealthy individuals.

The following ALM models have been applied in Pension Funds: Cees Dert (1995) wrote his PhD on ALM model for a Dutch pension fund that includes chance constraints. Boender et al. (1998) developed and described the ORTEC model. The ALM system for Towers Perrin-Tillinghast by Mulvey (2000) has three components: a scenario generator (CAP:Link), an optimization simulation model (OPT:Link), and a liability- and financial-reporting module (FIN:Link). Kouwenberg (2001) focuses on comparing scenario generation methods for a multistage stochastic programming model of a Dutch pension fund. InnoALM by Geyer (2006) is an ALM model for the largest corporate pension plan in Austria of Siemens Österreich.

One of the areas in which ALM is widely accepted is the banking industry. In particular, ALM models for banks should be created in line with the Basel II accord. Pioneering work includes Kusy and Ziemba (1986), where they introduce a multi-period stochastic linear programming ALM model for the Vancouver City
and Savings Credit Union. The goal is to maximize expected income minus expected costs. The drawback is that the model is not truly dynamic and the way of describing uncertainties is kept simple. Oguzsoy and Güvers (1997) present a multiperiod stochastic linear ALM model with simple recourse for a Turkish bank. It is important to meet regulations and policies while allowing withdrawals of depositors at any time. Banks are affected by credit, liquidity, capital and interest rate risk.

A rather new application of ALM is models for hedge and mutual funds. A detailed introduction is given in Ziemba (2003).

A wide application field of ALM is for Insurance Companies: one of the most famous implemented models for insurance companies is the Russell-Yasuda Kasai model (1991); it is a large-scale multiperiod stochastic programming model. The objective is to maximize discounted expected wealth minus discounted expected penalty costs, while meeting regulatory requirements, keeping enough cash reserves and offering competitive insurance policies. Consigli and Dempster (1998) designed the Computer-aided asset/liability management (CALM) model, which maximizes terminal wealth at the end of the time horizon. The Watson model is a specific instance of the CALM model for a pension fund. Hoyland et al. (2001) analyze the implications of regulations in the Norwegian life insurance companies using a multistage stochastic ALM model. Their results show that some regulations (the annual guaranteed rate of return for example) do not coincide with the insurance holders’ best interest. Frangos et al. (2004) propose a discrete time model of an insurance company incorporating transaction costs, non linear financial constraints and feasible portfolio control strategies. Within the rebalancing constraints, the monetary value of assets sold is equal to the monetary value of assets bought plus transaction costs. Rudolf and Ziemba (2004) presented a continuous time four-fund capital asset pricing model. The associated asset returns and liability returns followed an Ito process as functions of a risky state variable. The Prometeia model by Consiglio (2005) is an example of ALM for an Italian insurance company.

Further ALM models have been created for University Endowment Funds: in Mer-
ton (1991) the author sets out that universities offer education, training, research and storage of knowledge and that all of these activities incur a specific cost. University inflows consist of business income, property and grants. The corresponding ALM problem has been formulated and solved as a stochastic dynamic programming model.

ALM for wealthy individuals and families has been extensively explained by Ziemba (2003). Especially if high taxes apply and the income depends only on assets, a stochastic programming model might reallocate the assets while reducing the risk of unfavorable outcomes.

1.3 Guided Tour

The next section reviews the basic concepts of stochastic programming, chance constrained and integrated chance constrained programming, dynamic programming and dynamic control. Section 3 introduces a case study of an asset liability management problem for a pension fund. Section 4 considers scenario generating methods for both assets and liabilities. A linear programming based liability matching model is introduced in Section 5. The extension of this model to a stochastic setting is given in Section 6 as a stochastic programming recourse model, where in Section 7 we extend it to a chance and integrated chance-constrained setting. New recourse variables are included in a final model in Section 9 where the pension fund can invest in a bond index. Finally, we discuss the results and provide conclusions.

2 Models for Asset and Liability Management (ALM)

The deterministic models for asset allocation decisions such as expected value linear programming models (EVLP) obtained using fixed mix approaches, see Mulvey
(2000), do not capture the random behaviour of the assets and liabilities. In this section we consider thus alternative approaches which are gaining progressive acceptance in computational models for decision making under uncertainty.

2.1 Stochastic Programming (SP)

Stochastic Programming is an established approach to optimum decision making under uncertainty. Stochastic Programming models combine the paradigm of dynamic linear programming with modelling of random parameters (scenario generation). The solution of such Stochastic Programming models leads to optimal decisions which hedge against future uncertainties. A Stochastic Programming model is based upon an event tree for the key random variables (Figure 1). Each node of the event tree has multiple successors, in order to model the process of information being revealed progressively through time. The stochastic Programming approach determines the optimal decision for each node of the event tree, given the information available at that point.

![Event Tree](image)

Figure 1: Event Tree

Altenstedt et al. (2003) combine stochastic linear programming with parameterized
policies to a hybrid approach, which constructs a policy function from a number of stochastic programming solutions, and used it for an ALM model for a Swedish life insurance company. The stochastic linear programming (SLP) model fluctuates with the scenario tree chosen to represent random variables; the advantage of a parameterized policy is that it is much more transparent as the decision rule is explicitly available. Their techniques can be used for a larger asset set without increasing the scenario tree size. By applying the hybrid approach the problems of simulating the solutions of a stochastic programming based approach are reduced and the results are easier to interpret, without sacrificing performance. The major drawback of using the hybrid approach is that both, the SLP based solution and the policy based solution have to be implemented. Hence the hybrid policies may more appropriately be seen as a heuristic to improve an SLP-based model.

Hilli et al. (2005) present a stochastic programming ALM model for a Finnish pension insurance company. For modelling the stochastic factors they used the model developed by Koivu et al. (2003), where seven economic factors are modelled using a Vector Equilibrium Correction model. Assets considered were cash, bonds, equities, property and loans to policyholders. The liability cash flows were affected by the development of pensioners’ salaries and population dynamics. Statutory constraints have been introduced to satisfy the statutory restrictions for Finnish pension insurance companies. They form not only a part of the decision variables, but they are also included directly in the objective function.

2.2 Chance Constrained Programming (CCP) and Integrated Chance Constrained Programming (ICCP)

Another important class of stochastic programming models is represented by the chance constraint problems (CCP). The first time chance-constrained programming was introduced was in 1958 by Charnes and Cooper. The asset and liability management approach has been extended by C. Dert (1995) to include binary variables to count the number of times a certain event happens. This feature has been used
to formulate chance constraints that are based on the probability distribution of states of the world that follows from the scenarios. The chance constraint can be written as an inequality such as

\[ P(h(x, \xi) \geq 0) \geq \beta, \]  

(1)

where \( P \) means "probability", \( x \) is the decision vector, \( \xi \) is the random vector and \( \beta \in \{0, 1\} \) is the reliability level.

For asset and liability management, this property is used to model and restrict the probability of underfunding, both at the planning horizon, as well as at intermediate points in time. Instead of defining the value of assets to be greater or equal to the funding at time \( t \), the solvency requirement can be formulated as a chance constraint. Therefore, the model can account for any probability distribution that can be reflected by the scenarios; the probabilities of underfunding are endogenous to the model and probabilities of underfunding are taken into account explicitly, at intermediate points in time, as well as at the planning horizon.

Let \( A_t \) be the asset value and \( L_t \) the liability values at time \( t \). Then the chance constraints are included by:

\[ M\delta_{t+1}^s \geq \alpha L_{t+1}^s - A_{t+1}^s \quad \forall s, t = 1...T - 1 \]  

(2)

\[ M(1 - \delta_{t+1}^s) - \frac{1}{M} \geq A_{t+1}^s - \alpha L_{t+1}^s \quad \forall s, t = 1...T - 1 \]  

(3)

\[ \sum_{s=1}^S \pi_s \delta_{t+1}^s \leq 1 - \beta_{t+1}, \quad t = 1...T - 1 \]  

(4)

\[ \delta_{t+1}^s \in (0, 1), \quad t = 1...T - 1 \]  

(5)

where \( M \) is an arbitrarily big number, \( \delta_{t+1}^s \in (0, 1) \) is a binary decision variable and \( \alpha \) is a user-specified level of meeting the liabilities. These additional constraints will be connected to the two-stage stochastic programming model to solve the pension fund ALM.
Integrated chance constraints programming (ICCP) has been introduced by Han-
eveld (1986) and it is a quantitative alternative to chance-constrained program-
m. Haneveld et al. (2005) discuss integrated chance constraints in their role
of short-term risk constraints in a strategic ALM model for Dutch pension funds.
The dynamic problem is set up as a multistage recourse model. The ALM problem
is to select decisions on allocations of the assets, the contributions and indexation
of future payments (relative to e.g. wage inflation). For the pension fund problem
they study in this paper, long-term solvency goals go together with short-term
constraints on the funding ratio, defined as the ratio of assets over (discounted)
future liabilities. Such restrictions are called integrated chance constraints. They
are closely related to the well-known Conditional Surplus-at-Risk (CSaR: a vari-
ant of Conditional Value-at-Risk (CVaR), see Haneveld (2006)) concept used in
financial applications. In an integrated chance constraint the shortage is measured
with respect to some a priori chosen threshold parameter, whereas in a conditional
surplus-at-risk constraint the threshold is equal to the surplus-at-risk, which is it-
self an outcome of the optimization process. There are two types of constraints:
actuarial principles and policies.

Following notation by Haneveld and van der Vlerk (2006) integrated chance con-
straints are defined by

$$
\mathbb{E}_\omega [\eta_i(x, \omega)] \leq \kappa_i, \quad \kappa_i \geq 0 \tag{6}
$$

which corresponds to individual chance constraints and

$$
\mathbb{E}_\omega [\max_{i \in I} \eta_i(x, \omega)] \leq \kappa, \quad \kappa \geq 0 \tag{7}
$$

as an alternative for a joint chance constraint. The shortfall parameter \(\kappa_i\) and \(\kappa\)
is a user defined maximum allowed expected shortfall. In an ALM problem where
the assets value is defined to be \(A\) and the liabilities value \(L\) we can include the
integrated chance constraint by:

$$
\mathbb{E}[(A_{t+1} - \alpha L_{t+1})(t, s)] \leq \kappa_t \tag{8}
$$
2.3 Dynamic Programming (DP) and Dynamic Control

Dynamic programming makes use of backward recursion to move from the last period to the first period in a backward manner. It was introduced by Bellman in 1957. The basic functional equation is:

\[ f_N(x) = \max_{0 \leq x_N \leq x} [g_N(x_N) + f_{N+1}(x - x_N)], \text{ for } N = 1, 2, ..., T - 1; \ x \geq 0 \]

(9)

This is a recurrence relation connecting \( f_N(x) \) and \( f_{N+1}(x) \). It also satisfies Bellman’s principle of optimality stating the following: “An optimal policy has the property that whatever the initial state and initial decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.”

Daellenbach and Archer (1969) used stochastic dynamic programming to solve a bank ALM model with two assets and one liability and illustrated an example for a time span of ten days divided into five periods. Adachi (1996) combined stochastic linear programming with stochastic dynamic programming to solve a multiperiod portfolio problem with two assets and four periods. He used Benders decomposition with Monte Carlo sampling to decompose the problem into smaller subproblems. While stochastic linear programming problems grow exponentially in the number of periods, dynamic programming problems increase by size linearly in the number of periods. Musumeci and Musumeci (1999) incorporate individual’s aversion of risk to find the optimal asset allocation. The approach is especially useful for retirement asset allocation. They determine the optimal asset allocation for one year prior to retirement and roll backwards in time using dynamic programming to determine the optimal asset allocation for two years, three years, and so on prior to retirement. Their findings show that young investors shall be more aggressive and invest heavily in equities. Dupacova et al. (2002) compare modelling and solution of discrete time decision problems under uncertainty using multistage stochastic programming and dynamic programming. For dynamic programming a finite number of states is assumed and the aim is to find the optimal decision rule, while in stochastic programming there is no need for state definitions.
and their formulation is not connected to a prescribed solution method. Infanger (2006) applied stochastic dynamic programming and stochastic programming to asset allocation problems. An investor can choose between different asset classes, i.e. equities or fixed income, to build his portfolio. He showed that dynamic asset allocation with rebalancing periods leads to superior results than static or myopic techniques, i.e. Markowitz risk-return models.

Stochastic control problems capture uncertainty by allowing for a continuum of states which can be described at a given point in time by a smaller number of state variables that follow a joint Markov process. Compared to stochastic programming, the size of the problem grows exponentially with the number of state variables. Brennan et al. (1997) use stochastic control to solve a portfolio problem with three asset classes. They found out that the investor’s time horizon does affect the asset allocation of the optimal portfolio. Furthermore, their results suggest that a tactical asset allocation (TAA) for pension fund portfolio surpluses (the liabilities are mimicked by a duration matched bond portfolio) will not result in an optimal strategy for a long planning horizon.

3 A Case Study in ALM

We consider a pension plan ALM problem. The pension fund has a stream of liabilities, which they wish to meet by using a fixed income portfolio. The aim is to find an appropriate allocation that matches the liabilities while being affordable. Bradley and Crane (1972) introduced a bond portfolio problem using a dynamic decision-tree approach with decision variables being buying, selling and holding of bonds. Zenios et al. (1998) show that multi-stage stochastic programming models for fixed income portfolio management under uncertainty outperform portfolio immunization and single period models.

The assets and liabilities of the model are described in the next subsections.
3.1 Asset Classes

Since pension funds need to be in accordance to the FRS17 regulation, which uses a discount rate on AA rated corporate bond yields, the main asset allocation will be in fixed income instruments. By adopting this strategy expected volatility in the balance sheet can be significantly reduced. Bond classes include government, corporate, municipal, sovereign and agency bonds. Furthermore, for hedging options interest rate swaps can be included. An interest rate swap is an agreement between two parties, which exchange regularly interest-rate payments. The most common swap is a plain-vanilla interest rate swap, where one party agrees to make payments at a fixed rate, while the other party agrees to make floating rate payments.

3.2 Liabilities

Each pension fund has a series of cash flows it expects to pay out in the future to its existing members. Future payments depend on the longevity and possible earlier retirement of the member, salary growth rates, wage inflation and price inflation. The world’s populations are aging, having more pensioners and having more pensioners living longer, due to improved living standards and medical improvements. Pension fund valuation assumes a closed member population, new members are not accounted for because they don’t have a right to claim anything against the fund yet. The pension funds net cash is the contribution from members and employers minus benefit payments and operating costs. Pension plan funds estimate their liability stream for a time horizon up to 100 years given in yearly intervals. The current value of the cash flow stream is affected by the interest rate. For an introduction to pension mathematics, see Winklevoss (1993).

3.3 Representation of time

Since we are dealing with a long time horizon, time intervals can be represented as $N$ time buckets. A time bucket considers of several time points. For a precise
PV01 matching, the initial time buckets might only consider a short time period, i.e. only a few months, while the last time buckets might consider a longer time period, for example five years. Figure 2 shows that representation of time.

![Figure 2: Representation of time](image)

4 Scenario Models for Assets and Liabilities

The assets and their respective asset prices which we consider are for fixed income instruments and derivatives. We present some well known models for asset and liability pricing, while we only focus on one method for our computational results. The liabilities are random, too, where the exact nature is captured in scenarios meeting regulations and longevity. Vasicek and the CIR models are term structure models describing the evolution of the zero curve over time; the Heath, Jarrow, and Morton Model uses forward rates to an existing term structure of interest rates to determine asset prices. Section 4.1-4.3 give asset pricing models and Section 4.4 gives a model for a pension fund liability.
4.1 The Heath, Jarrow, and Morton Model

A general framework for pricing interest rate derivatives is the Heath, Jarrow, and Morton (1992) model. The interest rate movement can be described with today’s forward short rates. The instantaneous forward rate \( f(t, T) \) for a fixed maturity \( T \) evolves in a way which is described by the following diffusion process

\[
df(t, T) = \alpha(t, T)dt + \sigma(t, T)dW(t),
\]

\[
f(0, T) = f^M(0, T),
\]

with \( T \mapsto f^M(0, T) \) the market instantaneous-forward curve at time \( t = 0 \), and where \( W \) is a Brownian motion, \( \sigma(t, T) \) is a vector of adapted processes and \( \alpha(t, T) \) is itself an adapted process. The function \( \alpha \) cannot be arbitrarily chosen, it must equal a quantity depending on the vector volatility \( \sigma \) and on the drift rates in the dynamics of \( N \) selected zero-coupon bond prices. If the dynamics (2) are under the risk-neutral measure, then

\[
\alpha(t, T) = \sigma(t, T) \int_t^T \sigma(t, s)ds = \sum_{i=1}^N \sigma_i(t, T) \int_t^T \sigma_i(t, s)ds,
\]

such that

\[
f(t, T) = f(0, T) + \int_0^t \sigma(u, T) \int_u^T \sigma(u, s)ds \, du + \int_0^t \sigma(s, T)dW(s)
\]

\[
= f(0, T) + \sum_{i=1}^N \int_0^t \sigma_i(u, T) \int_u^T \sigma_i(u, s)ds \, du + \sum_{i=1}^N \int_0^t \sigma_i(s, T)dW_i(s)
\]
Therefore, we get the following zero-coupon bond price \( P(t, T) \) and the short term interest rate \( r(t) \) at time \( t \):

\[
dP(t, T) = P(t, T)[r(t)dt - \int_t^T \sigma(t, s)ds]dW(t),
\]

where

\[
r(t) = f(t, t) = f(0, t) + \int_0^t \sigma(u, t) \int_u^t \sigma(u, s)ds du + \int_0^t \sigma(s, t)dW(s)
\]

\[
= f(0, t) + \sum_{i=1}^N \int_0^t \sigma_i(u, T) \int_u^t \sigma_i(u, s)ds du + \sum_{i=1}^N \int_0^t \sigma_i(s, T)dW_i(s)
\]

(14)

### 4.2 The Vasicek Model

The Vasicek model is an One-Factor equilibrium model, where the risk-neutral Itô process of the short-term interest rate \( r \) is

\[
dr = a(b - r)dt + \sigma dz,
\]

where \( a, b \) and \( \sigma \) are constants. The model can have a negative short-term interest rate \( r \) and it is mean reverting, such that the short rate is pulled towards \( b \).

The price of a zero coupon bond at time \( t \), maturing at \( T \) is:

\[
P(t, T) = A(t, T)\exp(-B(t, T)r(t)),
\]

where

\[
B(t, T) = \frac{1 - e^{-a(T-t)}}{a}
\]

and

\[
A(t, T) = e^{\left[\frac{B(t, T)(a^2b - \sigma^2/2) - \sigma^2B(t, T)^2}{4a}\right]}
\]

(15-18)

### 4.3 The Cox, Ingersoll, and Ross Model

Another One-Factor equilibrium model is the Cox, Ingersoll, and Ross model (CIR), where the risk neutral process of the short-term interest rate is

\[
dr = a(b - r)dt + \sigma\sqrt{r}dz,
\]

(19)
where \( a, b \) and \( \sigma \) are constants. Unlike the Vasicek model, the short-term rate \( r \) will be always positive.

The price of a zero coupon bond is

\[
P(t, T) = A(t, T)e^{B(t, T)r(t)},
\]

where

\[
B(t, T) = \frac{2(e^{\gamma(T-t)} - 1)}{(\gamma + a)(e^{\gamma(T-t)} - 1) + 2\gamma}
\]

and

\[
A(t, T) = \frac{2\gamma e^{(a+\gamma)(T-t)/2}}{(\gamma + a)(e^{\gamma(T-t)} - 1) + 2\gamma}^{2ab/\sigma^2}
\]

4.4 Future Pension Cash Flows Projection

Pension fund cash outflows have been studied extensively, see for example textbooks (Winklevoss (1993) and Bowers (1997)) or research papers (see Winklevoss (1982)). We present our findings in a summary form. A pension fund’s cash outflow consists of payments to disabled and retired members and a lump sum payment to widows or orphans. From Figure 3 the new employees and the active employees represent the active workforce, who pay into the pension fund. While the insured employees, i.e. disabled or retired member and their families are receiving the cash outflows of the pension fund.

In order to generate the outflows we need to know (1) how many participants are in the closed fund population and (2) how much they are or they were earning. Plan participants can have the following statuses: active (A), retired (R), disabled (D), terminated (T) or dead (M). The most obvious reason for mortality is the person’s age, but also other factors can be taken into account, such as gender (males tend to die earlier than females), the occupation (different jobs have different stress exposure) and residential country.

The composite survival function gives the probability that an active member survives the next year, while keeping his active status: Let \( q^{(k)} \) be the probability of decrement and \( q'^{(k)} \) the rate of decrement, then the probability of an active member
where \( p_x \) is the probability to survive until the \( x + 1 \)th year. In order to get the corresponding \( q^{(k)} \), \( q'^{(k)} \) and \( p_x \) standard UK life tables are used.

The closed fund population model
A closed fund population considers only members until today, new joining members are not considered since they are not having any right on today’s pension fund. It is assumed that the age of entry into the plan is 21 and that all members retire with 65. There are no distinction between female and male and for simplicity we do not consider disability. The closed pension fund population is a deterministic survivorship group (cohort). Let \( l_x \) be the initial group in the population aged \( x \) and all members are subject to decrement specified by \( q^{(k)} \) from UK life tables.
The set of lives attaining age \( x \) in the survivorship group are:

\[
l_1 = l_0 p_0 \\
l_2 = l_1 p_1 = (l_0 p_0) p_1 \\
l_3 = l_2 p_2 = (l_0 p_0 p_1) p_2 \\
\vdots \\
l_x = l_{x-1} p_{x-1} = l_0 \prod_{y=0}^{x-1} p_y = l_0 x p_0
\]  

(24)

Following pension mathematics we have the following notation: \( x p_0 = \prod_{y=0}^{x-1} p_y \), where \( x p_0 \) is the effective \( x \)-year rate of survival, starting at year 0. The total liability profile is made up of the retired member liability profile plus the active member liability profile. Figure 4 shows an example of a typical liability profile in a defined benefit pension scheme:

The total expected cash outflows in a defined-benefit pension scheme is made up of two components: (a) the expected retirement benefit cash outflows for current active members who will retire in one or more years and (b) the expected cash outflows for today’s retired members. The first graph shows the typical active member liability profile (a), the second graph the typical retired member liability profile (b) and the last graph (c) shows the combination of both.

**Salary Curves**

The member’s retirement benefit is linked to their salary. Therefore, we need future estimates of the employee’s salary, which is influenced by (1) salary increase due to merit (seniority increases), (2) salary increase due to labor’s share of productivity and (3) increase due to inflation.

As employees gain more experience within their career, their contribution to the company increases which reflects merit increase. The merit increase decreases as the employee becomes older. We can therefore compute the salary function \( s_x \) for a participant at age \( x \):

\[
s_x = s_y \frac{(SS)^x}{(SS)^y} [(1 + I)(1 + P)]^{(x-y)}
\]

(25)
Figure 4: Defined-Benefit Liability Profiles
where

- $y$ is the entry age of the employee,
- $x$ is the age of the employee, $x > y$
- $s_y$ is the entry-age dollar salary,
- $(SS)_x$ is the merit salary scale at age $x$,
- $P$ is the rate of productivity,
- $I$ is the rate of inflation.

For simplicity we assume a constant merit and productivity rate and incorporate stochastic RPI (Retail Price Index) values to represent inflation rates in the UK.

For Figure 5 we assumed a productivity rate of 1% and an inflation rate of 3.5%. The merit salary scale data was taken from Winklevoss (1993). The expected per-
percentage increase in salary for different age-entries is plotted: for a member who entered with the age of 20, is expected to have a salary 19.5 times higher, where a member who entered by 60 will only have a salary 1.2 times higher when he/she retires.

**Forecasting Inflation**

Various models can be used to forecast inflation: Factor models, for example by Stock and Watson (1999), Vector Autoregression (VAR) models, Autoregressive Integrated Moving Average (ARIMA) or Neural Networks (NN), for example Nakamura (2005). Factor models incorporate a wide range of information about a time series; only a small number of unobserved factors influence the series. Other forecasting methods are based on the Phillips curve (mostly the unemployment rate Phillips curve). In our model ARIMA is used to generate inflation scenarios.

Autoregressive integrated moving average (ARIMA) models by Box and Jenkins (1976) combine autoregressive (AR) models with moving average (MA) models. Nonseasonal ARIMA models are defined as ARIMA($p,d,q$), where $p$ is the number of autoregressive terms, $d$ is the number of nonseasonal differences and $q$ is the moving average parameters.

The autoregressive model (AR) is as follows:

$$x_t = \xi + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \phi_3 x_{t-3} + \ldots + \varepsilon,$$  \hspace{1cm} (26)

where $\xi$ is a constant and $\phi_1, \phi_2, \phi_3, \ldots$ are the autoregressive model parameters. Hence, $x_t$ is a combination of the random error component $\varepsilon$ and a linear combination of prior observations $x_{t-1}, x_{t-2}, \ldots$.

The moving average model (MA) is as follows:

$$x_t = \mu + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \theta_3 \varepsilon_{t-3} - \ldots,$$  \hspace{1cm} (27)

where $\mu$ is a constant and $\theta_1, \theta_2, \theta_3$ are the moving average parameters. Hence, $x_t$ is a combination of the random error shock $\varepsilon$ and a linear combination of prior random shocks.
The general autoregressive integrated moving average process (ARIMA) is:

\[ W_t = \alpha_1 W_{t-1} + ... + \alpha_p W_{t-p} + Z_t + ... + \beta_q Z_{t-q} \]  

(28)

An ARIMA(1,1,2) process is fitted to get inflation rate scenarios for the UK.

5 Mathematical Model (1): Ex ante decision by LP

We consider an ALM method which uses an immunization technique: PV01 matching. PV01 is the change in the net present value (NPV) of a bond due to a basis-point (0.01%) positive parallel shift in the yield curve. PV01 matching attempts to match the interest rate sensitivity of the bond portfolio with the interest rate sensitivity of the liability stream. Our assets are a set of bonds \( B \) and our time horizon is \( t = 1, ..., T \).

5.1 Linear Programming (LP) based matching

The following linear programming model represents the PV01 matching approach mentioned above. The indices are the time buckets \( t = 1, ..., T \) and bond \( b \) from a set of bonds \( B \). In reality \( b \) is a function of the quadruplet \( i, j, k, l \) that is guaranteed by the set \( \{(i, j, k, l) \mid i \in \text{Ratingclass}, j \in \text{Sectors}, k \in \text{Country}, l \in \text{Issuer}\} \). Decisions are made at the beginning of the new time point.

The model indices are:

- \( b \): bond from a bond set \( B \)
- \( t \): time horizon \( 1, ..., T \)
Secondly, we define the parameters:

- $L_t$: liability value at time $t$
- $LPV01_t$: PV01 value of the liability at time $t$
- $BPV01_b^t$: PV01 value of bond $b$ at time $t$
- $c_b^t$: cash flow stream of bond $b$ at time $t$
- $P_b$: trading price of bond $b$
- $\alpha$: transaction cost involved when bonds are sold or bought
- $O_b$: opening position of bond $b$ at the initial time period
- $r_t$: interest rate at time $t$

Now we define the decision variables. All variables are constrained to be nonnegative.

- $x_b$: amount of bond $b$ purchased
- $y_b$: amount of bond $b$ sold
- $z_b$: amount of bond $b$ held
- $d_t$: amount borrowed at time period $t$; limited to be $\leq M_t \forall t$
- $s_t$: amount lent at time period $t$
- $C$: initial available cash

Furthermore, we use the following decision variables to measure the deviations of the assets $PV01$:

- $devo_t$: measurement of over deviation at time $t$
- $devu_t$: measurement of under deviation at time $t$

The set of constraints include some basic constraints for portfolio optimization: these are the cash-flow accounting equation and the inventory balance constraints. The cash-flow accounting equation initializes the new asset portfolio by liquidating the old asset allocation and including the available cash $C$. 

25
\[
\sum_{b=1}^{B} (1 + \alpha) P_b x_b \leq C + \sum_{b=1}^{B} (1 - \alpha) P_b y_b
\]  
(29)

The inventory balance constraint assures that the holding of bond \( b \) is made up of the opening position of the same bond including the amount bought minus the amount sold:

\[
z_b = O_b + x_b - y_b \quad \forall \ b
\]  
(30)

The following constraints match the bonds cash flows with the liabilities stream, having the possibility to borrow from the bank or reinvest spare cash. At the final time period there is no opportunity to borrow from the bank anymore.

\[
\sum_{b=1}^{B} c^b_1 z_b + d_1 = L_1 + s_1
\]  
(31)

\[
\sum_{b=1}^{B} c^b_t z_b + d_t - (1 + r_t) d_{t-1} = L_t + s_t - (1 + r_t) s_{t-1} \quad \forall \ t : 2 \leq t \leq T
\]  
(32)

\[
\sum_{b=1}^{B} c^b_T z_b - (1 + r_T) d_{T-1} = L_T + s_T - (1 + r_T) s_{T-1}
\]  
(33)

Furthermore, we define the behavior of the variables \( devo_t \) and \( devu_t \) with the following constraint that binds them to the difference of the assets and liabilities \( PV01 \) values:

\[
\sum_{b=1}^{B} BPV01^b z_b = devo_t - devu_t + LPV01_t \quad \forall \ t
\]  
(34)

If we include the minimum and maximum amount of bonds in the portfolio within a specific rating class, say AAA rating, specified by the pension fund, we get the
following user specific constraints:

\[
\text{MinInRatingclassAAA} \leq \sum_{\text{bonds}} z_{t(i=\text{AAA},...)}^{b(i=\text{AAA},...)} \leq \text{MaxInRatingclassAAA} \quad \forall \; t \quad (35)
\]

Similar limits are imposed in respect of other bond characteristic, that is country, issuer and sector.

Equation (27) is also called a goal constraint, where any under- and overdeviations are penalized in the objective function.

5.2 Two Objective Functions

The decision model we consider has two objective functions and the direction of optimization is minimization. The pension fund has to operate efficiently and this is achieved by keeping its contributions (employees plus employers) low. An increased need of initial cash means an increase in the active members’ contributions. The first objective function is the initial cash, which has to be injected in advance to achieve a (feasible) matching between assets and liabilities:

\[
\phi_1(d_t) = \text{Minimize } C + \sum_{t=1}^{T} d_t \quad (36)
\]

The second objective function is the total deviations of assets and liabilities; there are other suitable objective functions we might consider depending on the decision values required.

The objective function minimizes the total \( PV01 \) deviations of assets and liabilities. This linear objective function is given by

\[
\phi_2(\text{devo}_t, \text{devu}_t) = \text{Minimize } \sum_{t=1}^{T} (\text{devo}_t + \text{devu}_t). \quad (37)
\]
5.3 Computing the Efficient Frontier

The resulting bond allocation of the $PV01$ matching method only hedges against small shifts in the interest rate. The Pareto optimal frontier also known as the efficient frontier provides a trade-off between the two objective functions. In order to construct the efficient frontier we solve the model for minimizing initial cash and for minimizing total $PV01$ asset and liability deviations. We then force the resulting solution and solve with respect to total $PV01$ deviations and initial cash, respectively. For example, we find the minimum initial cash, then we use the result to solve for the minimum of $PV01$ deviation and vice versa. Then we take equi-distant points in between, i.e. different initial cash and total $PV01$ deviations, force these values and solve again. Figure 6 illustrates the computed efficient frontier for both methods. The red points give the initial cash values for equally spaced $PV01$ deviation points and the blue points give the $PV01$ deviations for initial cash funding increasing linearly. It is clear from the graph that for large values of initially injected cash, the total asset and liability $PV01$ deviations decrease slowly, while too little initial cash increases the deviations rapidly.

The AMPL problem formulation and summary results are in the Appendix.

6 Mathematical Model (2): Ex ante decision by Stochastic Programming

Stochastic Programming is a well known technique for ALM applications, but it is only being slowly introduced in the financial industry. One major issue is the size of the resulting optimization problem: the size of the deterministic equivalent problem grows exponentially with the number of scenarios which leads to computationally challenging decision problems. But this can be overcome (a) by recent developments in the computing field with increasing computing power and (b) through improvement in solution algorithms, which can process larger models (see Sen (1991) or Fabian).
Figure 6: PV01 Deviation-Budget Trade Off
The second argument for using a stochastic programming model for the liability determined investment problem is the planning horizon. We consider a time horizon of up to 100 years. The deterministic model only considers a small shift in the term structure of interest rates, which is highly unrealistic for a large considered time period.

6.1 Stochastic Programming (SP) Model

We consider a two-stage stochastic programming model. The event tree shown in Figure 7 represents the problem. We introduce an index set of scenarios $S$, each scenario represents possible future outcomes and with each scenario we also associate a probability $\pi_s$ of occurrence. The model will determine an optimal decision by using past information and anticipating future events. Uncertainty is only resolved between Stage 1 and Stage 2.

![Two Stage Event Tree](image)

Figure 7: Two Stage Event Tree

We have the same indices as the first model plus an additional set, which represents the scenario:
The following parameters are affected by uncertainty:

\(L_t^s\) liability value at time \(t\) under scenario \(s\)
\(r_t^s\) uncertain interest rate at time \(t\)
\(\pi_s\) probability of scenario \(s\) occurring

We replace the \(PV01\) notation by using present values:

\(LPV_t^s\) uncertain present value of the liability at time \(t\)
\(BPV_{t,b,s}^s\) uncertain present value of bond \(b\) at time \(t\)

The following are the stochastic decision variables:

\(d_t^s\) uncertain amount borrowed at time period \(t\); limited to be \(\leq M_t\) \(\forall t\)
\(PVd_t^s\) present value of the uncertain amount borrowed at time period \(t\); limited to be \(\leq M_t\) \(\forall t\)
\(s_t^s\) uncertain amount lent at time period \(t\)
\(PVs_t^s\) present value of the uncertain amount lent at time period \(t\)
\(devo_t^s\) measurement of over deviation at time \(t\) at scenario \(s\)
\(devu_t^s\) measurement of under deviation at time \(t\) at scenario \(s\)

Decisions are made before each time period. The restrictions (constraints) are comparable to the expected value linear programming (EVLP) problem described earlier. Wherever appropriate uncertain parameters and decision variables are included. Thus, the two-stage model is:

Cash-flow accounting equation:
\[
\sum_{b=1}^{B} (1 + \alpha) P_b x_b \leq C + \sum_{b=1}^{B} (1 - \alpha) P_b y_b
\]  

(38)

**Inventory balance equation:**

\[
z_b = O_b + x_b - y_b \quad \forall \ b
\]  

(39)

**Matching equations for all time periods:**

\[
\sum_{b=1}^{B} c_i^b z_b^b + d_i^s = L_i^s + s_i^s \quad \forall \ s
\]  

(40)

\[
\sum_{b=1}^{B} c_i^b z_b^b + d_i^s - (1 + r_t^s) d_i^{s-1} = L_i^s + s_i^s - (1 + r_t^s) s_i^{s-1} \quad \forall \ s, t \geq 2
\]  

(41)

\[
\sum_{b=1}^{B} c_i^T z_b^T - (1 + r_T^s) d_i^{s-1} = L_T^s + s_i^s - (1 + r_T^s) s_i^{s-1} \quad \forall \ s
\]  

(42)

**Present value matching of assets and liabilities:**

\[
\sum_{b=1}^{B} BPV_t^{b,s} z_b^s + PV_s = devo_t^s - devu_t^s + LPV_t^s + PV d_t^s \quad \forall \ s, t
\]  

(43)

**Non-Anticipativity constraints:**

\[
d_i^s = d_i^s \quad s = 2, ..., S
\]  

(44)

\[
s_i^s = d_i^s \quad s = 2, ..., S
\]  

(45)
Equation (36) is also called a goal constraint with stochasticity. Any over- and underdeviations are penalized in the objective function. Nonanticipativity constraints (Equation (37) and (38)) are required to ensure that the sets of feasible decisions depend only on past decisions and observed realizations of stochastic data and not on the future scenario realizations. Therefore, state variables need to take the same decisions if they share the same node in a scenario tree.

6.2 Two Objective Functions

Similar to the first model, there are two objective functions and the direction of optimization is minimization. Taking into consideration all possible future outcomes, the first objective function is the injected initial cash to meet future liabilities.

\[
\phi_3(d^*_t) = \text{Minimize } C + \sum_{t=1}^{T} \sum_{s=1}^{S} \pi_s d^*_t
\]  

(46)

The second objective function is the total deviations of bonds and liabilities present value and becomes

\[
\phi_4(\text{dev}_t^*, \text{devu}_t^*) = \text{Minimize } \sum_{t=1}^{T} \sum_{s=1}^{S} \pi_s (\text{dev}_t^* + \text{devu}_t^*)
\]  

(47)

6.3 Computing the Efficient Frontier

Again we plot the efficient frontier of the two stochastic programming objective functions. First, the minimum total deviations of assets and liabilities without restrictions on initial cash and the minimum initial cash with no restrictions on total asset/liability deviations is computed. Again equidistant cash and deviations
points are fixed in the model, that is then solved for the first and second objective function respectively.

In Figure 8 the red points give the deviation values for equidistant initial cash values and the blue points give the initial cash values for equally spaced deviation. From the graph we see a convex efficient frontier: lower initial cash values increase the deviation and vise versa.

The AMPL program and summary results are in the Appendix.

6.4 The Annotated Sampl Model

The stochastic programming model has been formulated in SAMPL that is an extended version of AMPL designed to express and generate stochastic models:
A scenarios index is introduced into the model, as the interest rates and the discount rates are indexed over this dimension:

\[ \text{scenarioset scenarios;} \]

The scenarios in the ALM model follow a discrete uniform probability distribution; this is expressed with the following:

\[ \text{probability param prob\{scenarios\}:}=1/\text{card\{scenarios\);} \]

The scenario-dependent parameter:

\[ \text{random param int\_rate\{timecf,scenarios\}>=0;} \]
\[ \text{random param discount\{timecf,scenarios\};} \]
\[ \text{random param disc\{scenarios,timecf,timecf\};} \]

The scenario tree considered in all our models is a two-stage tree, where the randomness is resolved after the first time period. The tree section is therefore:

\[ \text{tree theTree:= twostage\{card\{scenarios\}\);} \]

The staging information, that specifies the recursive actions:

\[ \text{suffix stage LOCAL;} \]

\[ \text{var bdbuyamnt\{b in bonds\}>=0, suffix stage 1;} \]
var bdsellamnt{b in bonds : bdopen[b]>0}>=0, suffix stage 1;

var bdholdamnt{b in bonds : bdopen[b]>0}>=0, suffix stage 1;

var availcash>=0, suffix stage 1;

let {t in timecf,s in scenarios} borrow[t,s].stage:= if (t=1) then 1 else 2;

let {t in timecf,s in scenarios} loan[t,s].stage:= if (t=1) then 1 else 2;

let {t in timecf,s in scenarios} deviateo[t,s].stage:= if (t=1) then 1 else 2;

let {t in timecf,s in scenarios} deviateu[t,s].stage:= if (t=1) then 1 else 2;

7 Mathematical Model (3&4): Ex ante decision by Chance-Constrained and Integrated Chance-Constrained Programming

For an introduction to Chance-Constrained Programming (CCP) see Section 2.2. We use a notation similar to Drijver and van der Vlerk. We define a reliability level \( \beta \), \( 0 \leq \beta < 1 \) for the chance constraints, which is the probability of satisfying a constraint. Then we introduce \( \gamma \), that is the level on which liabilities are to be met to satisfy the constraints. If \( \gamma = 1 \) we are trying to meet precisely the liabilities; if \( \gamma > 1 \) we force to have more inflows than outflows. The value of the liability at time \( t \) under scenario \( s \) is denoted by \( L_s^t \) and the value of the asset portfolio at time \( t \) under scenario \( s \) is denoted by \( A_t^s \).
This constraint applies for all time periods $t$ and for all scenarios $s$, which means that there is a chance constraint at every node in our scenario tree. The resulting scenario tree is shown in Figure 9; at each node, only a predefined number of child nodes can have underfunding. The ratio between the total number of scenarios branching from that node and the number of scenarios in which underfunding can occur is the specified $\beta_t$.

To formulate the chance constraints in the modelling language, we express them as:

$$P\{A_{t+1}^s - \gamma L_{t+1}^s \geq 0 \mid (t, s)\} \geq \beta_t \quad t = 1, \ldots, T - 1$$

(48)

Figure 9: Chance Constraint Scenario Tree

$$N\delta_{t+1}^s \geq \gamma L_{t+1}^s - A_{t+1}^s, \quad \forall s, t = 1, \ldots, T - 1 \quad (49)$$

$$\pi_s \sum_{s=1}^S \delta_{t+1}^s \leq 1 - \beta_t \quad \forall s, t = 1, \ldots, T - 1 \quad (50)$$

$$N(1 - \delta_{t+1}^s) - \frac{1}{N} \geq A_{t+1}^s - \gamma L_{t+1}^s \quad \forall s, t = 1, \ldots, T - 1 \quad (51)$$
where $N$ is a sufficiently large number and $\delta_t^s \epsilon \{0, 1\}$ is a binary variable, which takes on the value 0 if there is no underfunding, i.e. $\gamma L_t^s < A_t^s$. For $\beta_t$ at least one $\delta_t^s$ must be one to indicate underfunding. All scenarios are assumed to be equiprobable; every scenario occurs at the same probability: the probability of scenario $s$ occurring is $\pi_s = \frac{1}{S} \forall s$.

Another way of incorporating risk of underfunding is using Integrated Chance Constraints (ICC), introduced by Haneveld (1986), see also van der Vlerk (2006). The idea behind ICC is that not only the probability of underfunding is considered, but also how much the amount of underfunding is. The integrated chance constraint for our model is set out as follows:

$$
\pi_s \sum_{s=1}^{S} \delta_t^s (A_t^{s+1} - \gamma L_t^{s+1}) \leq \lambda L_t^s \hspace{1cm} \forall \ s, t = 1, \ldots, T - 1 \tag{52}
$$

$$
\delta_t^s \epsilon \{0, 1\} \hspace{1cm} \forall \ s, t = 1, \ldots, T - 1, \tag{53}
$$

where $\lambda$ is the maximum accepted expected shortfall in fraction of the liabilities and $\delta = 1$ if $A_t^s \leq \gamma L_t^s$.

The previous equations can also be represented in the following way, where we avoid the use of computationally expensive binary variables:

$$
A_t^s - \gamma L_t^s + \text{shortage}_t^s \geq 0 \hspace{1cm} \forall t, s \tag{54}
$$

$$
\pi_s \sum_{s=1}^{S} \text{shortage}_t^s \leq \lambda L_t^s \hspace{1cm} \forall t, s \tag{55}
$$

where $\text{shortage}$ is a variable that measures the amount of underfunding at time $t$ under scenario $s$. Integrated chance constraints are similar to conditional surplus-at-risk (CSaR), which is a variant of conditional value-at-risk (CVaR). Equation (48) is also called a global constraint.
7.1 Chance-Constrained and Integrated Chance-Constrained Model

The chance-constrained programming (CCP) model has the same sets, parameters, decision variables and constraints as the SP model discussed in the previous chapters:

The sets:

\begin{align*}
    b & \quad \text{bond from a bond set } B \\
    t & \quad \text{time horizon } 1, \ldots, T \\
    s & \quad \text{scenario } 1, \ldots, S
\end{align*}

The parameters:

\begin{align*}
    c^b_t & \quad \text{cash flow stream of bond } b \text{ at time } t \\
    P_b & \quad \text{trading price of bond } b \\
    \alpha & \quad \text{bid-ask spread in percentage when bonds are sold or bought} \\
    O_b & \quad \text{opening position of bond } b \text{ at the initial time period} \\
    \gamma_t & \quad \text{weight of liabilities with respect to the asset value at time } t; \text{ likely to be } > 1 \\
    \text{and then decreasing with time} \\
    \beta_t & \quad \text{reliability level at time } t; \text{ likely to be } > 0; \\
    \text{can be user defined or assumed to be constant over time} \\
    \lambda_t & \quad \text{maximum expected shortfall in fraction of the liabilities at time } t; \\
    \text{again user defined or assumed to be constant} \\
    N & \quad \text{sufficiently large number for the chance constraint} \\
    \text{(maximum value the investment portfolio is likely to reach)}
\end{align*}

The stochastic parameters:
$L_t^s$ liability value at time $t$ under scenario $s$
$r_t^s$ uncertain interest rate at time $t$
$\pi_s$ probability of scenario $s$ occurring, equiprobable
$LPV_t^s$ uncertain present value of the liability at time $t$
$BPV_t^{b,s}$ uncertain present value of bond $b$ at time $t$

The implementable first stage decision variables:

$x_b$ amount of bond $b$ purchased
$y_b$ amount of bond $b$ sold
$z_b$ amount of bond $b$ held
$C$ initial cash

The following are the non-implementable stochastic decision variables:

$d_t^s$ uncertain amount borrowed at time period $t$; limited to be $\leq M_t \forall t$
$s_t^s$ uncertain amount lent at time period $t$
$shortage_t^s$ amount of underfunding at time $t$
$\delta_t^s$ binary variable, which takes the value of 1 if there is any underfunding

The constraints:

*Cash-flow accounting equation:*

$$\sum_{b=1}^{B} (1 + \alpha)P_b x_b \leq C + \sum_{b=1}^{B} (1 - \alpha)P_b y_b \quad (56)$$

*Inventory balance equation:*

$$z_b = O_b + x_b - y_b \quad \forall b \quad (57)$$
Matching equations for all time periods:

\[
\sum_{b=1}^{B} c_b z_b + d_s^1 = L_s^1 + s_s^1 \quad \forall \ s
\]  

(58)

\[
\sum_{b=1}^{B} c_t^b z_b + d_t^s - (1 + r_t)d_{t-1}^s = L_t^s + s_t^s - (1 + r_t)s_{t-1}^s \quad \forall \ s, t \geq 2
\]  

(59)

\[
\sum_{b=1}^{B} c_t^b z_b - (1 + r_T)d_{T-1}^s = L_T^s + s_T^s - (1 + r_T)s_{T-1}^s \quad \forall \ s
\]  

(60)

Non-Anticipativity constraints:

\[
d_1^s = d_1^1 \quad s = 2, ..., S
\]  

(61)

\[
s_1^s = d_1^1 \quad s = 2, ..., S
\]  

(62)

The chance constraints are formulated as follows:

\[
P\left\{ \sum_{b=1}^{B} BPV_{b,t+1}^s z_b + PV s_{t+1}^s - \gamma LPV_{t+1}^s - \gamma PV d_{t+1}^s \geq 0 \mid (t, s) \right\} \geq \beta_t \quad \forall \ t
\]  

(63)

\[
N \delta_{t+1}^s \geq \gamma LPV_{t+1}^s + \gamma PV d_{t+1}^s - \sum_{b=1}^{B} BPV_{b,t+1}^s z_b - PV s_{t+1}^s \epsilon_t \leq 1 - \beta_t \quad \forall \ t
\]  

(64)

\[
\pi_s \sum_{s=1}^{S} \delta_{t+1}^s \leq 1 - \beta_t \quad \forall \ t
\]  

(65)

\[
N(1 - \delta_{t+1}^s) - \frac{1}{N} \geq BPV_{b,t+1}^s z_b + PV s_{t+1}^s - \gamma LPV_{t+1}^s - \gamma PV d_{t+1}^s \quad \forall \ t, s
\]  

(66)

\[\delta_t^s \in (0, 1) \quad \forall \ t
\]  

(67)
where the shortfall probability is measured for all scenario and is constrained at a reliability level. This is a direct way of modelling risk and incorporating risk aversion. For different time periods, different reliability levels $\beta$ can be set. It makes sense to set a high $\beta$ in the earlier time periods, while lowering the level of $\beta$ through time. The more distant the time period is from today, the more the world will probably differ from the forecasted scenarios; therefore in latter time periods it is very conceivable that the model will be re-run with the added knowledge of the world, and the planned actions changed accordingly. High reliability levels in early time periods still ensure the fund resulting in a surplus.

The integrated chance constraints can be included in the following way:

$$\sum_{b=1}^{B} BPV_{b,t} z_b + PV s_t - \gamma LPV_t - \gamma PV d_t + shortage_t \geq 0 \ \forall t, s \quad (68)$$

$$\pi \sum_{s=1}^{S} shortage_t \leq \lambda max_s (LPV_t) \ \forall t \quad (69)$$

The objective functions are:

$$\phi_5 (d_t^*) = \text{Minimize } C + \sum_{t=1}^{T} \sum_{s=1}^{S} \pi_s(d_t^*) \quad (70)$$

and:

$$\phi_6 (devo_t^*, devu_t^*) = \text{Minimize } \sum_{t=1}^{T} \sum_{s=1}^{S} \pi_s(devo_t^* + devu_t^*) \quad (71)$$

### 7.2 Computational Results

Figure 10 shows different $\beta$ values and the corresponding initial required cash. A constant $\gamma$ of 1.05 has been used and the chance constraints were only forced for the
first three time periods. As expected for a higher reliability level of $\beta$ more initial cash is required. The shape of the graph can be explained as follows: It is not a straight line, because of the relationship of $\beta$, $\delta$ and the total number of scenarios. If $\beta$ is changed my 0.1, then the sum of allowed $\delta$s in equation 58 will change by also 10% of the total number of scenarios; it is already as such "expensive" to assure that the probability of underfunding events is restricted, an increase of 10% in that probability might even be more expensive in terms of initial injected cash. In case of a low probability of total underfunding events, it might not be expensive to increase that probability. For example it is as expensive to have a $\beta$ of 0.75 as having a $\beta$ of 0.825, but there is a higher increase of initial cash from $\beta = 0.9$ to $\beta = 1$ than from $\beta = 0.8$ to $\beta = 0.9$.

Figure 11 compares the deviation-budget trade off of the stochastic programming model with the chance-constrained programming model with $\gamma = 1.05$ and $\beta = 0.875$ for three time periods. The thick green line is the efficient frontier of the

![Figure 10: CCP Beta-Budget Trade Off](image)
chance-constrained programming model and the blue points are the efficient frontier points of the stochastic programming model. The chance-constrained programming model requires more initial cash for the same deviations to assure that only a pre-specified percentage of scenarios is allowed to go underfunded. Due to the included binary variables the CCP takes more CPU time to solve; the generated model statistics can be found in the Appendix.

Our formulation of integrated chance constraints does not require the use of binary variables, which reduces the CPU compared to the CCP. Figure 12 shows the ICCP results in comparison with the SP results, with $\gamma = 1.05$ and $\lambda = 0.01$ for all time periods. The ICCP is more expensive in terms of initial injected cash for the same total deviations, but at the same time the deficit is only limited to be 1% of the maximum allowed liabilities.

Figure 11: SP versus CCP Deviation-Budget Trade Off
Figure 12: SP versus ICCP Deviation-Budget Trade Off
7.3 The Annotated Sampl Model

SAMPL has also language constructs that enable the user to incorporate chance constraints and integrated chance constraints into their stochastic model. The notation for the CCP and ICCP are as follows:

CCPConstraint\{t \text{ in time: } t>1\}:
\begin{align*}
\text{probability}\{s \text{ in scenarios:}
\sum_{b \text{ in bonds}} \text{bondpv}[b,t,s] \times \text{hold}[b]
+ (1+\text{interest}[t,s]) \times \text{loan}[t-1,s]
\geq \gamma \times (\text{liabilitypv}[t,s]
+ (1+\text{interest}[t,s]) \times \text{borrow}[t-1,s])
\}
\geq \beta[t];
\end{align*}

ICCPConstraint\{t \text{ in time: } t>1\}:
\begin{align*}
\text{expectation}\{s \text{ in scenarios}
\text{(} \gamma \times (\text{liabilitypv}[t,s]
+ (1+\text{interest}[t,s]) \times \text{borrow}[t-1,s]))
\text{less} \sum_{b \text{ in bonds}} \text{bondpv}[b,t,s] \times \text{hold}[b]
+ (1+\text{interest}[t,s]) \times \text{loan}[t-1,s]
\text{)}
\leq \lambda \times \text{MaxLiab}[t];
\end{align*}

This notation has several advantages over the use of standard mathematical programming languages: the user does not need to either include a new binary decision variable in the CCP case nor a new decision variable for the ICCP case. Only a reliability level and shortfall measure have to be defined.

8 Fixed Mix Approach

The last presented approach is a common asset allocation strategy, which we use to compare the previous models with. It is a simple decision rule of constant rebalanc-
ing at all time periods. Constant proportions of wealth are invested in government bonds, corporate bonds and quasi/foreign government bonds. For further dynamic strategies including constant mix (Fixed Mix) see Perold et al. (1988). Especially in flat, but oscillating markets (i.e. prices go down and then back up to the same level) fixed mix models outperform buy-and-hold strategies.

9 Enhancing the Models

The proposed models are enhanced further by introducing more recourse actions, that make use of the financial market by buying, holding or selling in a bond index. A bond index is traded in markets, where a fund manager can buy, hold or sell his position. In reality asset management sees an important aspect in rebalancing their portfolios when certain signals or drawdowns occur. A difficulty of holding a portfolio of bonds with different maturities is the uncertainty of the availability of similar bonds in the future; this can be overcome by the use of bond indices, which represent a class of bonds. Fund managers often use bond indices to compare their bond portfolio with, see for example Vassiadou-Zeniou et al. (1996) who track a bond index. The bond indices are classified into currency, type of issuer, maturity band of the issues and the rating of the bonds within the index. The Merrill Lynch Corporate and Collateral AA 10-Year+ Total Return Index denominated in UK pounds will be used in the enhanced models. Figure 13 shows possible future values (scenarios) of the bond index used for the following model. Thus, the use of rebalancing models are more flexible and at the same time more realistic.

The new objective functions are:
Figure 13: Bond Index Scenarios
\[ \phi_9(PVDev) = \text{Minimize} \sum_{t=1}^{T} \sum_{s=1}^{S} \pi_s PVDev_t^s \] (72)

\[ \phi_{10}(C) = \text{Minimize} C + \sum_{t=1}^{T} \sum_{s=1}^{S} \pi_s d_t^s \] (73)

where the first objective minimizes the total expected present value deviations and the second one minimizes the initial injected cash.

To introduce a new investment class of asset and making more dynamic recourse actions, new parameters and new decision variables are included:

The stochastic parameters:

\( I_t^s \) bond index trading price at time \( t \) under scenario \( s \)

\( O_i \) opening position of the bond index

The following are the recourse decision variables:

\( u_t^s \) uncertain amount bought of the bond index at time period \( t \)

\( v_t^s \) uncertain amount sold of the bond index at time period \( t \)

\( w_t^s \) uncertain amount held of the bond index at time period \( t \)

\( PVDev_t^s \) uncertain total present value deviations at time period \( t \)

With the appropriate non-anticipativity constraints for the recourse actions:

\[ u_t^s = u_t^1 \quad \forall \ s = 2, ..., S \] (74)

\[ v_t^s = v_t^1 \quad \forall \ s = 2, ..., S \] (75)

\[ w_t^s = w_t^1 \quad \forall \ s = 2, ..., S \] (76)
And the definition of PV deviations:

$$PV\text{Dev}^s_t = devo^s_t + devu^s_t \quad \forall \ t, s$$  \hspace{1cm} (77)

The following constraints further define the new recourse actions:

**Cash-flow accounting equation:**

$$\sum_{b=1}^{B} (1 + \alpha)P_b x_b + (1 + \alpha)I^s_1 u^s_1 \leq C + \sum_{b=1}^{B} (1 - \alpha)P_{by_b} + (1 - \alpha)I^s_1 v^s_1 \quad \forall \ s$$  \hspace{1cm} (78)

**Inventory balance equations:**

$$w^s_1 = O_I + u^s_1 - v^s_1 \quad \forall \ s$$  \hspace{1cm} (79)

$$w^s_t = w^s_{t-1} + u^s_t - v^s_t \quad \forall \ t \geq 2, s$$  \hspace{1cm} (80)

**Matching equations for all time periods:**

$$\sum_{b=1}^{B} c^b_t z_b + d^s_t - (1 + r_t)d^s_{t-1} + w^s_t I^*_t = L^s_t + s^s_t - (1 + r_t)s^s_{t-1} \quad \forall \ t \geq 2, s$$  \hspace{1cm} (81)

$$\sum_{b=1}^{B} c^b_T z_b - (1 + r_T)d^s_T - w^s_T I^*_T = L^s_T + s^s_T - (1 + r_T)s^s_{T-1} \quad \forall \ s$$  \hspace{1cm} (82)

**Present value matching of assets and liabilities:**
\[
\sum_{k=1}^{B} BPV_{t}^{h,s} z_{k} + PV s_{t}^{h} + PV (w_{t}^{h} I_{t}^{h}) = devd_{t}^{h} - devu_{t}^{h} + LPV_{t}^{h} + PV d_{t}^{h} \quad \forall s, t
\]

Figure 14 shows the deviation budget trade off of the enhanced SP, CCP and ICCP models. The ICCP model has \( \gamma = 1.05 \) and \( \lambda = 0.01 \) and the CCP has \( \beta = 0.875 \) and \( \gamma = 1.05 \). The SP and CCP models gain very similar results, while the ICCP has higher total deviations for the same initial cash of the SP and CCP. The focus on the ICCP is on over deviations which assure a possible limited underfunding.
10 Discussion and Conclusion

In this report, different ALM models for pension funds are introduced: two deterministic models and three stochastic models and their results are shown. The need for liability matching strategies using fixed income portfolios have been examined. Alternative asset pricing methods, which incorporate only interest rate uncertainties have been introduced into the models. Furthermore, methods of calculating a pension funds outflows are shown. Stochastic programming captures the uncertainty and the dynamic nature of pension fund problems and gives more realistic models than static fixed mix or $PV_01$ deviation matching models. The decisions obtained from the stochastic programming model dominate those obtained by the chance-constrained and the integrated chance-constrained programming models with respect to total present value deviations of assets and liabilities. An alternative feature of the chance-constrained and integrated chance-constrained programming models is that they give the user the possibility to include his risk aversion and directly model risk. Although the CCP and ICCP model have higher present value deviations, the focus is on overdeviations, which enables the pension fund to maintain excess return and limits the events of underfunding. Extensive evaluation and simulation including in-sample and out-of-sample testing is presented in a related report (see Schwaiger et al. (2008)). The models are written in AMPL (Fourer et al. 2003) and solved on a P4 3.0 GHz machine using ILOG CPLEX 10.1. The model statistics can be found in the Appendix. The CCP model takes by far the highest CPU time, followed by the ICCP and SP models. The number of non-zeros in the objective function can be determined by the size of the variables in the objective function. The LP objective function non-zeros are determined by the number of $PV01$ matching time periods, the SP objective function non-zeros are the number of over- and underdeviation matching time periods times number of scenarios (i.e. $T + T \times S$), the CCP objective function non-zeros are 1 plus number of time period of borrowing times number of scenarios (i.e. $1 + T \times S$) and the ICCP objective function non-zeros are the same as the SP’s. Similar calculations apply to the determination of the number of constraints and variables.
References


